ON THE ALGEBRAIC DIFFERENCE EQUATIONS $u_{n+2}u_n = \psi(u_{n+1})$ IN $\mathbb{R}_*^+$, RELATED TO A FAMILY OF ELLIPTIC QUARTICS IN THE PLANE

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We continue the study of algebraic difference equations of the type $u_{n+2}u_n = \psi(u_{n+1})$, which started in a previous paper. Here we study the case where the algebraic curves related to the equations are quartics $Q(K)$ of the plane. We prove, as in "on some algebraic difference equations $u_{n+2}u_n = \psi(u_{n+1})$ in $\mathbb{R}_*^+$, related to families of conics or cubics: generalization of the Lyness' sequences" (2004), that the solutions $M_n = (u_{n+1}, u_n)$ are persistent and bounded, move on the positive component $Q^0(K)$ of the quartic $Q(K)$ which passes through M_0, and diverge if M_0 is not the equilibrium, which is locally stable. In fact, we study the dynamical system $F(x,y) = ((a+ bx + cx^2)/y(c + dx + x^2), x)$, $(a,b,c,d) \in \mathbb{R}^{+4}$, $a + b > 0$, $b + c + d > 0$, in $\mathbb{R}_*^{+2}$, and show that its restriction to $Q^0(K)$ is conjugated to a rotation on the circle. We give the possible periods of solutions, and study their global behavior, such as the density of initial periodic points, the density of trajectories in some curves, and a form of sensitivity to initial conditions. We prove a dichotomy between a form of pointwise chaotic behavior and the existence of a common minimal period to all nonconstant orbits of F.

1. Introduction

In [4], we study the difference equations

$$u_{n+2}u_n = a + bu_{n+1} + u_{n+1}^2, \qquad u_{n+2}u_n = \frac{a + bu_{n+1} + cu_{n+1}^2}{c + u_{n+1}} \tag{1.1}$$

which generalize the Lyness' difference equations $u_{n+2}u_n = a + u_{n+1}$ (see [2, 7, 8, 9]). The first of these equations is related to a family of conics, and the second to a family of cubics (whose Lyness' cubics are particular cases). The results of [4] in the two cases are analogous to the results obtained in [3] about the global behavior of the solutions of Lyness' difference equation.

In the present paper, we will study the difference equation

$$u_{n+2}u_n = \frac{a + bu_{n+1} + cu_{n+1}^2}{c + du_{n+1} + u_{n+1}^2}. \tag{1.2}$$

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The dynamical system in ${\mathbb{R}_*^+}^2$ which represents this difference equation is

$$F(x,y)=\left(\frac{a+bx+cx^2}{y(c+dx+x^2)},x\right). \tag{1.3}$$

It is well defined as a homeomorphism of ${\mathbb{R}_*^+}^2$ when $a,b,c,d\geq 0$ and $a+b+c>0$, as we always assume. We have

$$M_{n+1}=(u_{n+2},u_{n+1})=F(M_n)=F(u_{n+1},u_n). \tag{1.4}$$

There is an invariant function

$$G(x,y)=xy+d(x+y)+c\left(\frac{x}{y}+\frac{y}{x}\right)+b\left(\frac{1}{x}+\frac{1}{y}\right)+\frac{a}{xy}, \tag{1.5}$$

which satisfies $G\circ F=G$, and thus $G(u_{n+1},u_n)$ is constant on every solution of (1.2).

If $K=G(u_1,u_0)$, the quartic $Q(K)$ with equation $G(x,y)=K$, or

$$x^2y^2+dxy(x+y)+c(x^2+y^2)+b(x+y)+a-Kxy=0, \tag{1.6}$$

passes through M_0.

The quartics $Q(K)$ are invariant on the action of F, and thus the points M_n move on the quartic passing through M_0, more precisely on its positive component $Q^0(K)$.

The map F has a geometrical interpretation. If $M\in{\mathbb{R}_*^+}^2$, let M' be the second point of the quartic $Q(K)$ which passes through M whose first coordinate is the same as those of M (there is only one such point M' because the point at the vertical infinity is a double point of the quartic). The image $F(M)$ is the symmetric point of M' with respect to the diagonal $x=y$.

For all this results, we refer to [4].

In Section 2, we give a general topological result useful for our study, which extends a result of [4], and we define a general property of weak chaotic behavior, whose proof for (1.2) is the goal of this paper.

In Section 3, we use this result to show that the solutions of difference equation (1.2) are, if $a+b>0$ and $b+c+d>0$, bounded and persistent in ${\mathbb{R}_*^+}^2$, and diverge if $(u_1,u_0)\neq(\ell,\ell)$, the fixed point of F, and prove that this point is locally stable.

In Section 4, we show that the case where $u_{n+2}u_n$ is a homographic function of u_{n+1}, studied in [5], comes down to our general model (1.2). This gives again, in a simpler way, results of [5], and improvements of them.

In Section 5, we study the case $a=0$, where the quartic passes through the point $(0,0)$. This case is easy, because a simple birational map transforms every quartic $Q(K)$ into a cubic curve studied in [4]. So we can apply the results of [4] without more work.

In Section 6, we prove general results in the case $a>0$, which lead to the fact that the restriction of the map F to each curve $Q^0(K)$ is conjugated to a rotation onto the circle (see Theorem 6.11). We study also in Sections 6 and 7 whether the chaotic behavior defined in Section 2 holds in the general case of (1.2), with a general property of dichotomy (see Theorem 6.18), and what happens in some particular cases (Section 7) and in the general one (Section 8).

In Section 9, we determine the possible periods of solutions of (1.2).

2. A topological tool for difference equations with an invariant

In this section, we give an abstract and more or less classical general result which will be useful for the study of difference equations. This assertion extends [4, Proposition 1].

Proposition 2.1. *Let X be a topological Hausdorff space. Let $F : X \to X$ and $G : X \to \mathbb{R}$ be two maps. Suppose first that the following conditions hold:*

(a) *F is continuous on X;*
(b) *G is continuous and has a strict minimum K_m at a point L;*
(c) *$\forall x \in X$, $G \circ F(x) = G(x)$ (the invariance property);*
(d) *F has at most one fixed point.*

If $K \geq K_m$, the level sets (if nonempty) of G are defined by $\mathscr{C}_K = \{x \in X \mid G(x) = K\}$. Then the following three results hold:

(1) *every point $x \in X$ lies in exactly one set $\mathscr{C}_K$;*
(2) *the point L is the (unique) fixed point of F;*
(3) *if $M_0 \in X$ let $M_{n+1} = F(M_n)$ be the points of the orbit of M_0 under F; then $M_n \in \mathscr{C}_{G(M_0)}$, and if $M_0 \neq L$, then the sequence (M_n) does not converge.*

Now suppose additional hypotheses:

(e) *X is connected and locally compact;*
(f) *$K_\infty := \lim_{x \to \infty} G(x) \leq +\infty$ exists, and $G < K_\infty$; then*
(4) *each $\mathscr{C}_K$ is compact and nonempty for $K_m \leq K < K_\infty$ (with $\mathscr{C}_{K_m} = \{L\}$), and the equilibrium point L is locally stable.*

Suppose at last the additional hypothesis:

(g) *G has only one local minimum (its global one at L); then*
(5) *for $K > K_m$ the set $\mathscr{C}_K$ is the boundary of the open set $U_K = \{G < K\}$ which is a connected relatively compact set.*

Proof. Assertions (1) and (2) are obvious. If $M_{n+1} = F(M_n)$, then $M_n \in \mathscr{C}_{G(M_0)}$. Suppose that M_n converges to a point N. Then $G(N) = G(M_0)$ and $F(N) = N$, so by (d) and (1) $N = L$. But $G(M_n) = G(N) = G(L) = K_m$, and by (b) $M_n = N$ for all n. Thus, if $M_0 \neq L$, then M_n does not converge.

If (e) and (f) hold, it is easy to see that $\mathscr{C}_K$ is nonempty and compact for every $K \geq K_m$; in particular, sequences (M_n) are bounded (i.e., relatively compact).

We prove now that the sets $U_K = \{G < K\}$ form a basis of neighborhoods of L. Let V be an open neighborhood of L. The sets $\{G \leq K\}$, for $K > K_m$, are compact, and their intersection is $\{L\}$; so there is a $K > K_m$ such that $\{G \leq K\} \subset V$, and thus $U_K \subset V$.

We can now prove easily that L is locally stable: if V is a neighborhood of L, there exists $K > K_m$ such that $U_K \subset V$. If $M_0 \in U_K$, then, for every n, $M_n \in U_K$ by (c), and $M_n \in V$.

We prove now assertion (5), if (g) also holds. We have $\overline{U_K} \subset \{G \leq K\}$, and $\overline{U_K} \setminus U_K \subset \{G \leq K\} \setminus \{G < K\} = \mathscr{C}_K$. Thus, $\partial U_K \subset \mathscr{C}_K$. Now, if $\mathscr{C}_K \not\subset \partial U_K$, there exists $x \in \mathscr{C}_K$, $x \notin \partial U_K$, thus there exists a neighborhood V of x such that $V \cap U_K = \emptyset$. Thus, $G \geq K$ on V, and $G(x) = K$; thus x is a local minimum of G, and $x \neq L$ because $K > K_m$: this is impossible, and $\partial U_K = \mathscr{C}_K$.

Finally, we prove that U_K is connected. If U_K is the union of two disjoint nonempty open sets A and B (which are relatively compact), put $\alpha = \inf_A G$ and $\beta = \inf_B G$; we have $\alpha, \beta < K$. If $\alpha = G(u)$ with $u \in A$ and $\beta = G(v)$ with $v \in B$, then u and v are two distinct local minima of G, which is impossible. Thus we can suppose that $\alpha = G(u)$ with $u \in \overline{A} \setminus A$. But $\overline{A} \subset X \setminus B$ (because U_K is open), thus $u \notin B$, and $u \in \overline{U_K} \setminus U_K = \partial U_K = \mathscr{C}_K$. So we have $G(u) = K > \alpha$, which is a contradiction. $\square$

We will use Proposition 2.1 when X is an open subset of $\mathbb{R}^2$; in this context, F and G are given by (2) and (3). In the general case, we can ask the question whether a form of chaotic behavior may be described for the map F (we will study in this paper if it is the case with F and G given by (2) and (3)). Precisely, one may ask the question whether F has an "invariant pointwise chaotic behavior," denoted IPCB.

Property of IPCB. We suppose that X, F, and G satisfy properties (a),(b),...,(g) of Proposition 2.1, and that X is a metric space, with distance d. We say that the dynamical system (X,F) with invariant G has IPCB if we have the following three properties.

(a) There exists a partition of $X \setminus \{L\}$ into two dense subsets A and B which both are union of "curves" $\mathscr{C}_K$, and then invariant under F : A is the set of initial periodic points M_0, B is the set of initial points M_0 whose orbit is dense in the curve $\mathscr{C}_K$ which passes through M_0 (that is $\mathscr{C}_{G(M_0)}$).

(b) Every point $M_0 \in X \setminus \{L\}$ has sensitivity to initial conditions, that is, there exists $\delta(M_0) > 0$ (this dependance on M_0 explains the term "pointwise") such that every neighborhood of M_0 contains a point M_0' whose iterates M_n' satisfy $d(M_n, M_n') \geq \delta(M_0)$ for infinitely many integers n.

(c) There exists an integer N such that every integer $n \geq N$ is the minimal period of some periodic orbit of F.

IPCB is the essential result of [3] about the behavior of Lyness' difference equation $u_{n+2}u_n = k + u_{n+1}$ if $0 < k \neq 1$ (if $k = 1$, 5 is a common minimal period to all nonconstant solutions).

In [4], we prove also that IPCB holds for the solutions of difference equations in $\mathbb{R}_*^{+2}$

$$u_{n+2}u_n = a + bu_{n+1} + u_{n+1}^2, \qquad u_{n+2}u_n = \frac{a + bu_{n+1} + cu_{n+1}^2}{c + u_{n+1}}. \tag{2.1}$$

An important tool to study the dynamical system linked to (1.2) may be an eventual property in the abstract case of Proposition 2.1: for every $K \in]K_m, K_\infty[$, is the dynamical system $F_{|\mathscr{C}_K}$ conjugated to a rotation on the circle with angle $2\pi\theta(K) \in]0,\pi[$? This eventual property supposes that each set $\mathscr{C}_K$ is homeomorphic to a circle. Then the study of the properties of function θ would be essential: continuity (analyticity if X is an open set of $\mathbb{R}^2$), limits when $K \to K_m$ and $K \to K_\infty$.

3. First general results of divergence and stability

We begin by identifying the fixed point.

Lemma 3.1. *If $a = b = 0$, then sequence (1.2) tends to* 0. *If $a + b > 0$, then the fixed point of the dynamical system (1.3) is the unique positive root ℓ of the equation*

$$Y^4 + dY^3 - bY - a = 0, \tag{3.1}$$

and it is the unique possible limit for sequence (1.2).

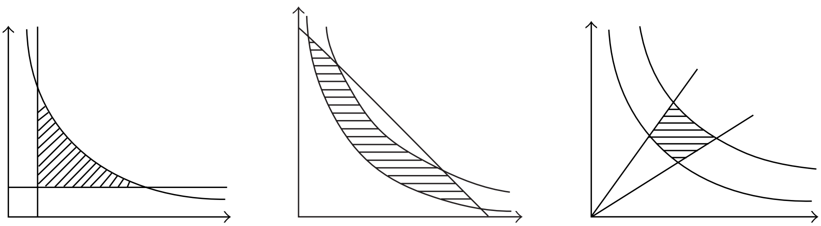

Figure 3.1

Proof. It is obvious that a fixed point of F has the form (Y,Y) where Y satisfies (3.1), and that (3.1) has a unique positive root ℓ such that (ℓ,ℓ) is invariant by F.

For the limit of the sequence (u_n) of (1.2), we must be more careful if $a = 0$, because in this case $Y = 0$ is solution of (3.1). But then $Q(K)$ passes through $(0,0)$ and has as tangent at this point the line $x + y = 0$ if $b > 0$, and so the point $M_n = (u_{n+1}, u_n)$, which lies on $Q(K) \cap \mathbb{R}_*^{+2}$, cannot tend to $(0,0)$.

If $a = b = 0$, the fixed point is solution of $Y^4 + dY^3 = 0$, which has no solution in $\mathbb{R}_*^+$. In this case, we have $u_{n+2}/u_{n+1} = (u_{n+1}/u_n)(c/(c + du_{n+1} + u_{n+1}^2))$, with $c > 0$ and $d \geq 0$. Thus $\rho_n = u_{n+1}/u_n$ is decreasing and tends to a limit λ. If $\lambda < 1$, then $u_n \to 0$. If λ would satisfy $\lambda \geq 1$, then u_n would be increasing, and would tend to infinity. But then $c/(c + du_{n+1} + u_{n+1}^2) \leq 1/2$ for big n, and thus we would have $\lambda = 0$, which is a contradiction. □

With the objective of using Proposition 2.1, it is necessary to study the function G. The first question is to know if $G(x,y) \to +\infty$ if (x,y) tends to the infinite point of the locally compact space $\mathbb{R}_*^{+2}$. It appears that this condition fails in the general case. Indeed, we look for a condition for the sets $A_K := \{G \leq K\} \cap \mathbb{R}_*^{+2}$ to be compact. The hypothesis is

$$xy + d(x+y) + c\left(\frac{x}{y} + \frac{y}{x}\right) + b\left(\frac{1}{x} + \frac{1}{y}\right) + \frac{a}{xy} \leq K. \tag{3.2}$$

Thus we have $xy + a/xy \leq K$, $d(x+y) \leq K$, $c(x/y + y/x) \leq K$, $b/x \leq K$, and $b/y \leq K$.

If $b > 0$, then $x \geq b/K$ and $y \geq b/K$, and thus, with the condition $xy \leq K$, the set A_K is compact. If $b = 0$, we will suppose $a > 0$ (the case $a = b = 0$ is trivial by Lemma 3.1), and the condition $xy + a/xy \leq K$ implies that $0 < r_1 \leq xy \leq r_2$: the point (x,y) is between two hyperbolas. But then if c or d is positive, we have $x/y + y/x \leq K/c$ and thus $0 < s_1 \leq y/x \leq s_2$, or $x + y \leq K/b$. In the two cases, A_K is compact; see Figure 3.1.

So the good condition in (1.2), which we suppose in all the sequel, is

$$a + b > 0, \qquad b + c + d > 0. \tag{3.3}$$

It is to be noticed that if $b = c = d = 0$, the function G does not tend to $+\infty$ at the point at infinity of $\mathbb{R}_*^{+2}$, and then the solutions of the difference equation (1.2) may be unbounded or not persistent. In fact, it is always the case.

Indeed, consider the difference equation $u_{n+2}u_n = a/u_{n+1}^2$, with $a > 0$. Its solutions are the sequences $u_n = a^{1/4}\exp[(-1)^n(A+Bn)]$, with $A = \ln(u_0 a^{-1/4})$ and $B = -\ln(u_0u_1a^{-1/2})$, which are neither bounded nor persistent if $u_0u_1 \neq \sqrt{a}$.

Then, we must identify the minimum of G. The equations of critical points are $x^2y^2 + dx^2y + c(x^2-y^2) - by - a = 0$ and $x^2y^2 + dy^2x + c(y^2-x^2) - bx - a = 0$. The difference of these two equations gives $(x-y)(dxy + 2c(x+y) + b) = 0$. But if $(x,y) \in \mathbb{R}_*^{+2}$, the only solution is $x = y$, so the previous equations give $x^4 + dx^3 - bx - a = 0$, and thus we have $x = y = \ell$: G has a unique critical point at the equilibrium $L = (\ell,\ell)$, the minimum of G is achieved only at this point, and the value of the minimum is

$$K_m = d\ell + 2c + \frac{3b}{\ell} + \frac{2a}{\ell^2}. \tag{3.4}$$

If $K > K_m$, then $Q^0(K) = Q(K) \cap \mathbb{R}_*^{+2} = \{(x,y) \in \mathbb{R}_*^{+2} \mid G(x,y) = K\}$ is a nonempty compact component of $Q(K)$, and through every point $M \in \mathbb{R}_*^{+2}$ passes a unique curve $Q^0(K)$. We can thus apply Proposition 2.1 and obtain the following theorem.

Theorem 3.2. *If $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $a+b > 0$, and $b+c+d > 0$, every solution of the difference equation (1.2)*

$$u_{n+2}u_n = \frac{a + bu_{n+1} + cu_{n+1}^2}{c + du_{n+1} + u_{n+1}^2} \tag{3.5}$$

is bounded and persistent in $\mathbb{R}_*^{+2}$. If $(u_1,u_0) \neq (\ell,\ell)$, then (u_n) diverges, the point $M_n = (u_{n+1},u_n)$ moves on the curve $Q^0(K)$ which passes through M_0, and $K > K_m$. Moreover the equilibrium L is locally stable.

4. The homographic case

In [5], the authors study the difference equation

$$u_{n+2} = \frac{\alpha u_{n+1} + \beta}{u_n(\gamma u_{n+1} + \delta)}, \quad \text{with } \alpha,\beta,\gamma,\delta \geq 0,\ \alpha+\beta > 0,\ \gamma+\delta > 0. \tag{4.1}$$

If $\gamma = \alpha = 0$, we find the classical sequence $u_{n+2} = (\beta/\delta)/u_n$ which is always 4-periodic.

If $\gamma = 0$, $\alpha \neq 0$, the sequence $v_n = (\delta/\alpha)u_n$ satisfies $v_{n+2}v_n = v_{n+1} + \beta\delta/\alpha^2$: it is a Lyness sequence, and its behavior is known and given in [3].

So, we suppose $\gamma > 0$, and thus we can suppose $\gamma = 1$.

Under this hypothesis, if we suppose that the two quadratic polynomials of (1.2) have a common root $x = -p < 0$, then (4.1) is a particular case of (1.2). To see this fact, we examine some cases.

(i) If $\delta \neq 0$, easy calculation shows that with

$$a = \frac{\alpha\beta}{\delta}, \qquad b = \frac{\alpha^2}{\delta} + \beta, \qquad c = \alpha, \qquad d = \frac{\alpha}{\delta} + \delta, \tag{4.2}$$

(1.2) is exactly (4.1) with $\gamma = 1$.

(ii) If $\delta = \alpha = 0$, (4.1) becomes $u_{n+2}u_n = \beta/u_{n+1}$, which is a classical 3-periodic sequence.

(iii) If $\delta = 0$, $\alpha > 0$, $\beta = 0$, $u_{n+2} = \alpha/u_n$ is another case of the previous 4-periodic sequence.

(iv) If $\delta = 0$, $\alpha > 0$, $\beta > 0$, we put $u_n = \beta/v_n$, and obtain $v_{n+2}v_n = \beta^2/(v_{n+1} + \alpha)$, which has the form $(\alpha' v_{n+1} + \beta')/(v_{n+1} + \delta')$. Thus, (1.2) for (v_n) with the values $a = c = 0$, $b = \beta^2$, $d = \alpha$, is exactly (4.1) for $u_n = \beta/v_n$.

In any cases, (4.1) comes down to (1.2) or to a known sequence (Lyness' one) or to elementary sequences (3- or 4-periodic). Thus, with the aid of elementary results on Lyness' equation (see [3]), we deduce again from Section 2 the result of [5] about (4.1), but almost without calculation, and we can improve it.

Proposition 4.1. *The solutions of (4.1) are bounded and persistent, and diverge if (u_1, u_0) is different than the fixed point. Moreover the equilibrium point is locally stable.*

Of course, other properties of the solutions of (4.1) will follow from the general property of solutions of (1.2) that we will prove in the following parts, see corollaries of Theorems 5.1 and 7.1, where examples of (4.1) which have IPCB are given.

5. The case $a = 0$

In this part, we solve the case when $a = 0$, which is simple, because an easy birational map reduces the associated quartic curves to cubic ones which give a previous case already solved (see [4]). So we obtain the following general result.

Theorem 5.1. *Let the difference equation in $\mathbb{R}_*^+$ be*

$$u_{n+2}u_n = \frac{bu_{n+1} + cu_{n+1}^2}{c + du_{n+1} + u_{n+1}^2} \quad \text{with } b > 0,\ c \geq 0,\ d \geq 0,\ c + d > 0, \tag{5.1}$$

whose solutions diverge if $(u_1, u_0) \neq (\ell, \ell)$. Let $L = (\ell, \ell)$ be the equilibrium, with ℓ positive solution of the equation $Y^3 + dY^2 - b = 0$. Let $F(x, y) = ((bx + cx^2)/y(c + dx + x^2), x)$ be the homeomorphism of $\mathbb{R}_*^{+2}$ associated to (5.1): $M_n := (u_{n+1}, u_n) = F^n(M_0)$. Let $Q_{b,c,d}(K)$ be the quartic curve with equation

$$x^2y^2 + dxy(x + y) + c(x^2 + y^2) + b(x + y) - Kxy = 0 \tag{5.2}$$

which passes through $M_0 = (u_1, u_0)$, and $Q^0_{b,c,d}(K)$ its positive component, globally invariant under the action of F.

(a) *There exists a well-defined number $\theta_{b,c,d}(K) \in]0, 1/2[$ such that the restriction of F to $Q^0_{b,c,d}(K)$ is conjugated to a rotation on the circle, of angle $2\pi\theta_{b,c,d}(K) \in]0, \pi[$.*

(b) *For every b, c, d satisfying the conditions of (5.1) and $b^2 \neq c^3$ or $bd \neq 2c^2$, the difference equation (5.1) has IPCB.*

(c) *Every integer $n \geq 4$ is the minimal period of some solution of (5.1) for some b, c, d, and some initial point M_0.*

One has $b^2 = c^3$ and $bd = 2c^2$ if and only if every solution of (5.1) is 5*-periodic.*

Proof. If $a = 0$, then the quartic curve (1.6) reduces to (5.2), and then it passes through $(0, 0)$.

(1) Case $c = 0$ and $d > 0$.
We define the birational map

$$X = \sqrt{\frac{b}{d}\frac{1}{x}}, \qquad Y = \sqrt{\frac{b}{d}\frac{1}{y}}, \tag{5.3}$$

that is the transformation on the solutions (u_n) of difference equation (5.1) by the formula

$$v_n = \sqrt{\frac{b}{d}\frac{1}{u_n}}. \tag{5.4}$$

Under map (5.3) the quartic (5.2) becomes the cubic of paper [4]:

$$\Gamma_\alpha(K') \quad \text{with } \alpha = \sqrt{\frac{b}{d^3}},\ K' = \frac{K}{\sqrt{bd}}, \tag{5.5}$$

associated to the difference equation

$$v_{n+2}v_{n+1}v_n = \alpha + v_{n+1} \tag{5.6}$$

whose solutions are studied in [4]. Then results of Theorem 5.1 are nothing else but [4, Proposition 8 and Theorem 4].

(2) Case $c > 0$.
We define now the birational map

$$X = \frac{b}{c}\frac{1}{x}, \qquad Y = \frac{b}{c}\frac{1}{y}, \tag{5.7}$$

that is the transformation on the solutions (u_n) of difference equation (5.1) by the formula

$$v_n = \frac{b}{c}\frac{1}{u_n}. \tag{5.8}$$

Under map (5.7) the quartic (5.3) becomes the cubic of (see [4])

$$\Gamma_{\alpha,\beta}(K') \quad \text{with } \alpha = \frac{b^2}{c^3},\ \beta = \frac{bd}{c^2},\ K' = \frac{K}{c}, \tag{5.9}$$

associated to the difference equation

$$v_{n+2}v_n = \frac{\alpha + \beta v_{n+1} + v_{n+1}^2}{v_{n+1} + 1} \tag{5.10}$$

whose solutions are studied in [4]. Then the results in Theorem 5.1 are nothing but [4, Proposition 11 and Theorem 6]. The case of 5-periodicity in [4] corresponds to the values $\alpha = 1$ and $\beta = 2$ (see [4, Lemma 8]), which gives the end of assertion (d) of the theorem.

Point (3) of [4, Theorem 6] has to be modified, because here we have only $\alpha > 0$ and $\beta \geq 0$ (arbitrary), but in [4] we have $\alpha \geq 0$ and $\beta > -2\sqrt{\alpha}$. So, in [4, Lemma 10], we must replace the domain D by $\widetilde{D} = \mathbb{R}^{+2}$ and the function $f(\ell) = (1/\pi)\cos^{-1}(1/2)(1 - 1/\sqrt{\ell+1})$ by the function $\widetilde{f}(\ell) = (1/\pi)\cos^{-1}(1/2)\sqrt{1+3/(\ell+1)}$. Then it is easy to show that we have only $\psi(\overset{o}{\widetilde{D}}) =]0,1/3[$. Thus every integer $n \geq 4$ is actually a period. □

Corollary 5.2. *The solutions of (4.1) studied in* [5], *where* $\alpha\beta\gamma > 0$, $\delta = 0$, *satisfy Theorem 5.1.*

Remark 5.3. (1) If $c > 0$, then solutions (u_n) of (5.1) are rational if and only if the v_n's are rational, when b, c, d are rational. Then, in this case, a rational periodic solution of (5.1) may have only periods which belong to the set $\{3,4,5,6,7,8,9,10,12\}$ (see [4]).

But if $c = 0$, the map (5.3) does not preserve rationality of real numbers, except if $b/d = q^2$ with $q \in \mathbb{Q}_*^+$. In this case, and with b rational, the periodic rational solutions of (5.1) may have only periods 7 or 10 (see [4, corollary of Proposition 7]).

(2) The 5-periodic case $b^2 = c^3$ and $bd = 2c^2$ corresponds to initial Lyness' sequence: $v_n = \sqrt{c}/u_n$ satisfies $v_{n+2}v_n = 1 + v_{n+1}$.

We give now two easy cases with $a = 0$, which are not covered by Theorem 5.1.

First, the case $a = b = 0$ is given by Lemma 3.1: the sequence tends to 0.

Second, we have the following classical result.

Lemma 5.4 (case $a = c = d = 0$, $b > 0$). *The positive solutions of the difference equation* $u_{n+2}u_{n+1}u_n = b$ *are 3-periodic.*

6. General results in the case $a > 0$

It is easy to see that if $a > 0$ we can suppose that $a = 1$ (put $u_n = v_n\sqrt[4]{a}$). We make this hypothesis from now on.

6.1. Points on the diagonal and the birational transformation of the quartic. We know that the quartic curve has two double points at infinity in vertical and horizontal direction, which are ordinary if $d^2 - 4c \neq 0$, the asymptotes being then the lines $x = r_1$, $x = r_2$, $y = r_1$, and $y = r_2$, where the r_i are the roots (real or complex) of the equation $s^2 + ds + c = 0$. Moreover, if $K > K_m$, the quartics $Q(K)$ have no singular point in $\mathbb{R}_*^{+2}$. Indeed, if the equation of $Q(K)$ is $p(x,y) - Kxy = 0$, singular points are given by $p'_x - Ky = 0$, $p'_y - Kx = 0$, and $p - Kxy = 0$. These relations give $xp'_x = p$ and $yp'_y = p$, and these last relations are the equations whose solutions are the critical points of the function $G(x,y) = p(x,y)/xy$. But we have seen that G has no critical point in $\mathbb{R}_*^{+2}$ except for $L = (\ell,\ell)$, and so the only finite singular point of $Q(K)$ in $\mathbb{R}_*^{+2}$ would be L, but this point is not on $Q(K)$ if $K > K_m$.

So we can hope that the quartic curve $Q(K)$ is an elliptic one, and thus that it can be transformed in a regular cubic curve by a birational transformation. To make such a transformation, some point of $Q(K)$ should disappear, and to preserve the symmetry of the curve with respect to the diagonal $\delta : x = y$, we choose this point on this diagonal. So the fundamental technical result will be the behavior of the points of $Q(K)$ on the diagonal.

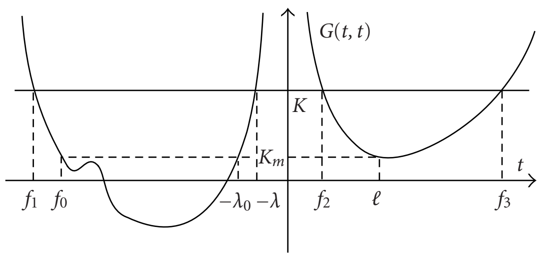

Figure 6.1

Lemma 6.1. *For $K > K_m$, the coordinates of the intersection points of $Q(K)$ with the diagonal δ are solutions of the equation*

$$t^4 + 2dt^3 + (2c - K)t^2 + 2bt + 1 = 0. \tag{6.1}$$

These coordinates are real numbers f_1, $-\lambda$, f_2, f_3 which satisfy

$$\begin{gathered} f_1 < -\ell < -\lambda < 0 < \lambda < f_2 < \ell < f_3 \quad \text{if } d + b > 0, \\ f_1 = -\frac{1}{\lambda} < -1 < -\lambda < 0 < \lambda = f_2 < 1 = \ell < f_3 = \frac{1}{\lambda} \quad \text{if } d = b = 0. \end{gathered} \tag{6.2}$$

Moreover, numbers f_i and λ are continuous functions of K on $]K_m, +\infty[$, whose limits when $K \to +\infty$ and $K \to K_m$ are

$$\lim_{K\to+\infty} \lambda = \lim_{K\to+\infty} f_2 = 0, \qquad \lim_{K\to+\infty} f_1 = -\lim_{K\to+\infty} f_3 = -\infty, \tag{6.3}$$

$$\begin{gathered} \lim_{K\to K_m} f_2 = \lim_{K\to K_m} f_3 = \ell, \qquad \lambda_0 := \lim_{K\to K_m} \lambda = (d+\ell) - \sqrt{(d+\ell)^2 - \frac{1}{\ell^2}}, \\ \lim_{K\to K_m} f_1 := f_0 = -(d+\ell) - \sqrt{(d+\ell)^2 - \frac{1}{\ell^2}}. \end{gathered} \tag{6.4}$$

Proof. Formula (6.1) is obvious. Let $h_K(t) = t^4 + 2dt^3 + (2c - K)t^2 + 2bt + 1 = 0$. By (3.1) and (3.4) we have

$$h_K(\ell) = d\ell^3 + 3b\ell + 2 + (2c - K)\ell^2 < d\ell^3 + 3b\ell + 2 - \ell^2\left(d\ell + \frac{3b}{\ell} + \frac{2}{\ell^2}\right) = 0, \tag{6.5}$$

$h_K(0) = 1$, and so h_K has two roots f_2 and f_3 which satisfy $0 < f_2 < \ell < f_3$. We have also $h_K(-\ell) = h_K(\ell) - 4d\ell^3 - 4b\ell < 0$, and thus we have two other roots f_1 and $-\lambda$ which satisfy $f_1 < -\ell < -\lambda < 0$. At last, $h_K(\lambda) = h_K(-\lambda) + 4d\lambda^3 + 4b\lambda = 4d\lambda^3 + 4b\lambda \geq 0$. If $b + d > 0$, thus we have $0 < \lambda < f_2$. If $b = d = 0$, the roots of h_K are f_1, $-\lambda$, λ, and $-f_1$, whose product is 1. This gives (6.2).

Then we remark that the equation $h(t) = 0$ is equivalent to the relation $G(t,t) = K$. But the graph of the function $t \mapsto G(t,t)$ is easy to determine, see Figure 6.1. It is immediate from this graph that the roots are continuous functions of K. Their limits when $K \to +\infty$

are obvious and given by (6.3). If $K \to K_m$, then f_2 and f_3 tend to ℓ, and f_1 and $-\lambda$ have limits f_0 and $-\lambda_0$ which are the two other roots of equation $h_{K_m}(t) = 0$, which has already the double root ℓ. Thus these two other roots are easy to obtain, they are given by (6.4). □

Now we write the equation of $Q(K)$ in the form:

$$X^2Y^2 + dXY(X+Y) + c(X^2+Y^2) + b(X+Y) + 1 - KXY = 0. \tag{6.6}$$

We make the birational transformation ϕ_K defined in affine coordinates, for $XY \neq \lambda^2$, by

$$x = \frac{X+\lambda}{XY-\lambda^2}, \qquad y = \frac{Y+\lambda}{XY-\lambda^2}, \qquad \text{or} \qquad X = \frac{1+\lambda x}{y}, \qquad Y = \frac{1+\lambda y}{x}, \tag{6.7}$$

or, in homogeneous coordinates, by

$$X' = x't' + \lambda x'^2, \qquad Y' = y't' + \lambda y'^2, \qquad T' = x'y', \tag{6.8}$$

or

$$x' = T'(X' + \lambda T'), \qquad y' = T'(Y' + \lambda T'), \qquad t' = X'Y' - \lambda^2 T'^2. \tag{6.9}$$

On $Q(K)$ this transformation ϕ_K is not defined only at the point $(-\lambda, -\lambda)$ if $d + b > 0$, and at the points $(-\lambda, -\lambda)$ and (λ, λ) if $b = d = 0$.

Now we determine the image of $Q(K)$ under ϕ_K. We substitute the second formulas of (6.7) in (6.6). Putting $D := 1 + \lambda(x+y)$, easy calculation gives for the left hand of the equation in variables x, y the product of D by the following factor

$$\begin{aligned}&(d\lambda^2 - c\lambda + b)xy(x+y) + \lambda c(x^3+y^3) + (c+d\lambda)(x+y)^2 + (d+\lambda)(x+y)\\ &\quad + (2\lambda^2 - 2d\lambda - 2c - K)xy + 1\end{aligned} \tag{6.10}$$

(the coefficient of x^2y^2 is $\lambda^4 - 2d\lambda^3 + (2c-K)\lambda^2 - 2b\lambda + 1$ which is 0 because the point $(-\lambda, -\lambda) \in Q(K)$).

So we obtain the straight line Δ_λ with equation $1 + \lambda(x+y) = 0$ and the cubic curve $\Gamma(K)$ with equation

$$(x+y)\big(\lambda c(x+y)^2 + \alpha(K)xy\big) + (c+d\lambda)(x+y)^2 + (d+\lambda)(x+y) - \beta(K)xy + 1 = 0, \tag{6.11}$$

where

$$\alpha(K) = d\lambda^2 - 4c\lambda + b, \qquad \beta(K) = K + 2c + 2d\lambda - 2\lambda^2. \tag{6.12}$$

With second formulas of (6.7), one sees that if $(x,y) \in \Delta_\lambda \setminus \{(-1/\lambda, 0), (0, -1/\lambda)\}$, then $(X,Y) = (-\lambda, -\lambda)$ which has no image by ϕ_K. Identification of the images of $Q(K) \setminus \{(-\lambda, -\lambda)\}$ and $Q^0(K)$ is given in the following results.

Lemma 6.2. (1) *If* $b+d>0$, *then* $XY>\lambda^2$ *on* $Q^0(K)$, *and if* $b=d=0$, *then* $XY\geq\lambda^2$ *on* $Q^0(K)$, *with equality only at the point* (λ,λ).

(2) *If* $b+d>0$, *then the positive component* $\Gamma^0(K)$ *of the cubic* $\Gamma(K)$ *is compact, and* ϕ_K *is a homeomorphism of* $Q^0(K)$ *onto* $\Gamma^0(K)$. *If* $b=d=0$, *then* $\Gamma^0(K)$ *is unbounded and has a point at infinity in direction* $x=y$, *which is the image by* ϕ_K *of the point* (λ,λ) *of* $Q^0(K)$, *and* ϕ_K *is a homeomorphism of* $Q^0(K)\setminus\{(\lambda,\lambda)\}$ *on* $\Gamma^0(K)$.

Proof. (1) We work here in $\mathbb{R}_*^{+2}$, and begin in the case when $b+d>0$. Suppose that there is a point (X,Y) in the set $\{G\leq K\}$, lying on the hyperbola $XY=\lambda^2$. Then we have $X+Y\geq 2\lambda$ and

$$\begin{aligned}0&\geq X^2Y^2+dXY(X+Y)+c(X+Y)^2-(2c+K)XY+b(X+Y)+1\\&\geq\lambda^4+2d\lambda^3+(2c-K)\lambda^2+2b\lambda+1\\&=h_K(-\lambda)+4d\lambda^3+4b\lambda\\&=4d\lambda^3+4b\lambda>0,\end{aligned}\tag{6.13}$$

and this is impossible. So the set $\{G\leq K\}$, which is connected by Proposition 2.1, is contained in one of the two connected components of $\mathbb{R}_*^{+2}\setminus\{XY=\lambda^2\}$. But $f_2>\lambda$ by Lemma 6.1, and thus $Q^0(K)\subset\{XY>\lambda^2\}$.

Now if $b=d=0$, we choose $(X,Y)\neq(\lambda,\lambda)$. Then $X+Y>2\lambda$ if $XY=\lambda^2$, and the same calculation gives, on $(\{G\leq K\}\setminus\{(\lambda,\lambda)\})\cap\{XY=\lambda^2\}$, the impossible inequality $0>0$. But this calculation proves also that $\{G<K\}\cap\{XY=\lambda^2\}=\varnothing$. So $\{G\leq K\}$ is contained in $\{XY\geq\lambda^2\}$ or in $\{XY\leq\lambda^2\}$, and we conclude, with the aid of the point (f_3,f_3), that $\{G\leq K\}\setminus\{(\lambda,\lambda)\}\subset\{XY>\lambda^2\}$.

(2) If $b+d>0$, we have $XY>\lambda^2$ on $Q^0(K)$, and formulas (6.7) show that ϕ_K is a homeomorphism of $Q^0(K)$ onto the positive component $\Gamma^0(K)$ of $\Gamma(K)$ (note that $\Gamma(K)$ does not intersect the axis $\{y=0\}\cap\{x\geq 0\}$ nor $\{x=0\}\cap\{y\geq 0\}$, by formula (6.11)). So the set $\Gamma^0(K)$ is compact in $\mathbb{R}_*^{+2}$.

If $b=d=0$, (6.7) shows that ϕ_K is a homeomorphism of $Q^0(K)\setminus\{(\lambda,\lambda)\}$ onto $\Gamma^0(K)$, and so $\Gamma^0(K)$ cannot be bounded. Equation (6.11) becomes

$$\lambda c(x-y)^2(x+y)+c(x+y)^2+\lambda(x+y)-\beta xy+1=0,\tag{6.14}$$

and this proves that $\Gamma(K)$ has the point at infinity in the direction of the diagonal. Moreover, (6.7) shows that when $(X,Y)\to(\lambda,\lambda)$ on $Q^0(K)$, then (x,y) tends to infinity in direction $x=y$ on $\Gamma^0(K)$. □

6.2. The algebraic-geometric interpretation of the transformed sequence, and the elliptic nature of the quartic and cubic curves. We begin with the transformation of the sequence $M_n=(u_{n+1},u_n)$ by ϕ_K: what is the behavior of the points $m_n=\phi_K(M_n)$?

Lemma 6.3. *Let* m_n *be the image in* $\Gamma^0(K)$ *of the sequence* $M_n=(u_{n+1},u_n)$ *on* $Q^0(K)$ *by the birational transformation* ϕ_K. *Then the symmetric point of* m_{n+1} *with respect to the diagonal* $x=y$ *lies on* $\Gamma^0(K)$ *and on the straight line* (P,m_n), *where* $P=(-1/\lambda,0)\in\Gamma(K)$ *(see Figure 6.2).*

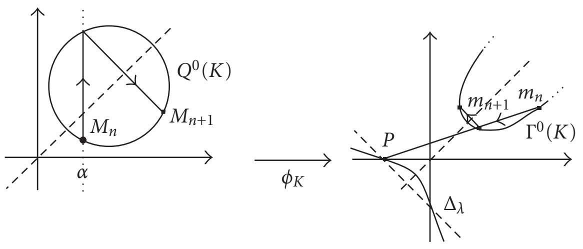

Figure 6.2

Proof. The symmetry with respect to the diagonal is preserved by ϕ_K. We look at the images by ϕ_K of the vertical lines $X = \alpha$. By (6.7), they are the straight lines with equations $\alpha y = 1 + \lambda x$, and all these lines pass through the point $P = (-1/\lambda, 0)$. So the lemma is obvious. □

Now we will, as in [3, 4], translate the property of sequence m_n given by Lemma 6.3 into the addition $m_{n+1} = m_n + P$ for a group law on the cubic curve $\Gamma(K)$. To do this, we need the regularity of the curve $\Gamma(K)$, that is its elliptic nature. With this objective, we make a new transformation κ, independant from K, and defined it by

$$\kappa(x, y) = (U, V), \quad \text{where } x + y = -U,\ x - y = V. \tag{6.15}$$

So $\Gamma(K)$ becomes a new cubic curve $\widetilde{\Gamma}(K)$ with equation

$$-U\left(\lambda c U^2 + \alpha(K)\frac{U^2 - V^2}{4}\right) + (c + d\lambda)U^2 - (d + \lambda)U - \beta(K)\frac{U^2 - V^2}{4} + 1 = 0, \tag{6.16}$$

or

$$V^2\bigl(\alpha(K)U + \beta(K)\bigr) - \bigl(b + d\lambda^2\bigr)U^3 + \gamma(K)U^2 - 4(d + \lambda)U + 4 = 0, \tag{6.17}$$

where

$$\gamma(K) = 4c + 4d\lambda - \beta(K) = 2\lambda^2 + 2d\lambda + 2c - K. \tag{6.18}$$

Now we need the following result.

Lemma 6.4. *For every $K > K_m$,*

$$\beta(K) > 4c + 3d\ell + \frac{b}{\ell} > 0. \tag{6.19}$$

Proof. It is obvious with formulas (3.1), (3.4), and (6.12), and condition (3.3). □

On the other hand, one can see that the quantity $\alpha(K)$ may be zero for some value of K, and positive or negative (see proof of Proposition 6.7). But if $4c^2 < bd$ (and thus $b > 0$ and $d > 0$), then $\alpha(K) \geq (bd - 4c^2)/d > 0$. In the general case of condition (3.3) only, we set

$$p(K) = \frac{\alpha(K)}{\beta(K)}. \tag{6.20}$$

Now we define a new transformation ψ_K, in affine coordinates, by

$$U = \frac{u}{1 - p(K)u}, \qquad V = \frac{v}{1 - p(K)u}, \tag{6.21}$$

or

$$u = \frac{U}{1 + p(K)U}, \qquad v = \frac{V}{1 + p(K)U}, \tag{6.22}$$

or, in homogeneous coordinates, by

$$U' = u', \qquad V' = v', \qquad W' = w' - p(K)u', \tag{6.23}$$

or

$$u' = U', \qquad v' = V', \qquad w' = W' + p(K)U'. \tag{6.24}$$

We obtain a new cubic curve $E(K)$.

Remark that if for some K one has $p(K) = 0$, then $\psi_K = Id$.

If $p(K) \neq 0$, then the line with equation $U = -1/p(K)$ is an asymptote of inflexion of $\widetilde{\Gamma}(K)$ which is sent to the line at infinity by ψ_K; this line is a tangent of inflexion to the cubic $E(K)$.

The equation of $E(K)$ is

$$\begin{aligned} v^2\beta = u^3&\left(b + d\lambda^2 + p\gamma + 4p^2(d+\lambda) + 4p^3\right) \\ &- u^2\left(12p^2 + 8p(d+\lambda) + \gamma\right) + u\left(4(d+\lambda) + 12p\right) - 4. \end{aligned} \tag{6.25}$$

Of course, if $p = 0$, we find again the cubic curve $\widetilde{\Gamma}(K)$ itself.

Now we can prove the essential result of this part.

Proposition 6.5. (1) *If $K > K_m$, the cubic curve $E(K)$ is regular: it is an elliptic curve. So there is on $E(K)$ an abelian group law whose unit element is the point at infinity in vertical direction, and whose addition is defined by $A + B + C = 0$ if and only if the three points A, B, C of $E(K)$ are on the same straight line (and the opposite $-A$ is the symmetric of A with respect to the u-axis).*

(2) *Let $\widetilde{m}_n$ be the images of points m_n by $\psi_K \circ \kappa$, and $\widetilde{P}$ the image of P by the same map. Then, for every n, $\widetilde{m}_{n+1} = \widetilde{m}_n + \widetilde{P}$, and $\widetilde{m}_n = \widetilde{m}_0 + n\widetilde{P}$, for the addition of the group law on $E(K)$.*

Proof. Equation of $E(K)$ has the form $\beta v^2 = P_3(u)$, with $\beta > 0$ and $\deg(P_3) \leq 3$. So we know (see [1, 6]) that $E(K)$ is regular if and only if

(i) the coefficient of u^3 in P_3 is nonzero ($\deg(P_3) = 3$);

(ii) the three roots of P_3 are distinct.

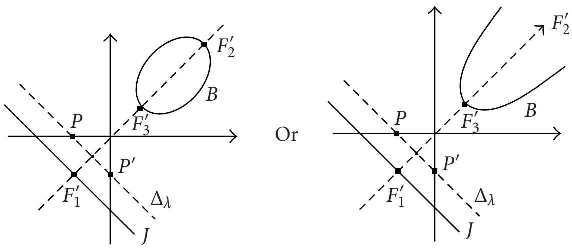

Figure 6.3

We prove first (i). If $p(K)=0$, then $E(K)=\widetilde{\Gamma}(K)$ whose equation is

$$\beta(K)v^2=\left(b+d\lambda^2\right)U^3-\gamma(K)U^2+4(d+\lambda)U-4. \tag{6.26}$$

The coefficient of U^3 vanishes only if $b=d=0$, but in this case $\alpha(K)=-4c\lambda\neq 0$, and $p(K)\neq 0$. So, if $p(K)=0$, then $b+d\lambda^2>0$.

Suppose then that $p(K)\neq 0$ and that the coefficient of u^3 in P_3 is zero. Then $E(K)$ splits into a conic and the line at infinity, that is $\widetilde{\Gamma}(K)$ splits into its asymptote $U=-1/p(K)$ and a conic (besides one see easily that the left hand of (6.17) has the factor $U+1/p(K)$). So, it is also the case of $\Gamma(K)$: it is the union of the real line J with equation $x+y=1/p(K)$ and a conic B. But $\Gamma(K)$ is symmetric with respect to the diagonal, contains the points $P=(-1/\lambda,0)$, $P'=(0,-1/\lambda)$, and the three distinct points $F_i'=(1/(f_i-\lambda),1/(f_i-\lambda))$, $i=1,2,3$, except if $b=d=0$, when F_2' is the point at infinity in direction $x=y$ (see Lemma 6.2). One has $F_2'\notin J$ and $F_3'\notin J$ (because $\Gamma(K)\cap\{x\geq 0\}\cap\{y=0\}=\emptyset$), so $F_1'\in J$.

If $b+d>0$, $\Gamma^0(K)$ is compact and contains F_2 and F_3 (see Lemma 6.2), so $\Gamma^0(K)=B$ is necessarily an ellipse, but in this case the real points P and P' cannot lie on $\Gamma(K)$ (because they do not belong to J: $F_1\notin\Delta_\lambda$), and this is a contradiction (see Figure 6.3).

If $b=d=0$, $\Gamma^0(K)$ is necessarily a parabola with axis $x=y$, and the same contradiction holds.

Now we prove point (ii). The three roots of P_3 are distinct. These roots are the first coordinates of the images of the (f_i,f_i) by the transformation $\psi_K\circ\kappa\circ\phi_K$. So on the coordinates we have the transformations $t\mapsto 1/(t-\lambda)$, $t\mapsto -2t$, $t\mapsto t/(1+p(K)t)$. From the numbers $f_1<0<f_2<f_3$, we obtain first $f_1'<0<f_3'<f_2'\leq+\infty$, and then $-\infty\leq\widetilde{f}_2<\widetilde{f}_3<0<\widetilde{f}_1$. If $p(K)\neq 0$, the images of these last numbers by $t\mapsto t/(1+p(K)t)$ are finite (P_3 has degree exactly 3), so $\widetilde{f}_i\neq -1/p(K)$ for $i=1,2,3$, and thus we obtain images $\widetilde{e}_1$, $\widetilde{e}_2$, $\widetilde{e}_3$ which are distinct. If $p(K)=0$, then $b+d>0$ and $\widetilde{f}_2>-\infty$, and $\widetilde{e}_i=\widetilde{f}_i$ are distinct. So point (1) of the proposition is proved.

Finally, the transformation $\psi_k\circ\kappa$ is projective, so it preserves the alignment of points; thus point (2) of the proposition is obvious from Lemma 6.3. $\square$

Corollary 6.6. *If $K>K_m$ and $a>0$, then the quartic curve $Q(K)$ is elliptic.*

PROPOSITION 6.7. (1) *The coefficient of u^3 in (6.25) is positive:*

$$q(K) := b + d\lambda^2 + p\gamma + 4p^2(d+\lambda) + 4p^3 > 0. \tag{6.27}$$

(2) *The roots $\widetilde{e}_i$ of P_3, images of the $\widetilde{f}_i$ by ψ_K, satisfy the inequalities*

$$\widetilde{e}_2 < \widetilde{e}_3 < \widetilde{e}_1. \tag{6.28}$$

Proof. We have seen that $q(K)$ is never zero, and that the three numbers $\widetilde{e}_i$ are distinct. We will use an argument of continuity and connectedness. Let Ω be the subset of $\mathbb{R}^3$ defined by $b \geq 0$, $c \geq 0$, $d \geq 0$, and $b+c+d > 0$; it is a convex set. We consider $K_m = 2c + d\ell + 3b/\ell + 2/\ell^2$ as a continuous function of $(b,c,d) \in \Omega$ (it is easy to see that ℓ is a continuous function on Ω). Let Σ be the subset of $\mathbb{R}^4$ defined by

$$\Sigma = \bigcup_{(b,c,d)\in\Omega} \{(b,c,d)\} \times \left]K_m(b,c,d), +\infty\right[, \tag{6.29}$$

that is the strict epigraph of K_m. It is easy to see that Σ is a connected set. So q, as continuous function on Σ, has a constant sign. But the function p vanishes for some value of $(b,c,d,K) \in \Sigma$, because if $c = d = 0$ and $b > 0$, then $\alpha = b > 0$, and if $b = d = 0$ and $c > 0$, then $\alpha = -4c\lambda < 0$: p must vanish at some point by the intermediate value theorem. But in this point q assumes the value $b + d\lambda^2 > 0$ because if $p = 0$, then $b + d > 0$. So we have $q > 0$ on Σ.

For the same reason, the functions $\widetilde{e}_i - \widetilde{e}_j$ do not vanish ($i \neq j$), and for $p = 0$ we have $\widetilde{e}_i = \widetilde{f}_i$ and $\widetilde{f}_2 < \widetilde{f}_3 < \widetilde{f}_1$: these inequalities hold also for $\widetilde{e}_i$. □

For a later use of Proposition 6.5, we look at the coordinates of the point $\widetilde{P}$. Easy calculations give its coordinates:

$$\widetilde{P} = \left(\frac{1}{p+\lambda}, -\frac{1}{p+\lambda}\right), \quad \text{with } p+\lambda \neq 0, \tag{6.30}$$

because P does not belong to the asymptote of $\Gamma(K)$, so $\widetilde{P}$ is a finite point of $\widetilde{\Gamma}(K)$.

We need the position of the coordinates of $\widetilde{P}$.

LEMMA 6.8. *The coordinates of the point $\widetilde{P}$ satisfy the inequalities*

$$\frac{1}{p+\lambda} > \widetilde{e}_1 > 0, \qquad -\frac{1}{p+\lambda} < 0. \tag{6.31}$$

Proof. First, on the set Σ one has $1/(p+\lambda) \neq \widetilde{e}_1$: otherwise, we would have $1/\lambda = \widetilde{f}_1 = -2f_1' = 2/(\lambda - f_1)$, and then $f_1 = -\lambda$, which is false by Lemma 6.1. Then, if $p = 0$, $\widetilde{e}_1 = \widetilde{f}_1$, and the inequality $1/(p+\lambda) = 1/\lambda > \widetilde{e}_1$ is equivalent to $2/(\lambda - f_1) < 1/\lambda$, or $f_1 < -\lambda$, which is true by Lemma 6.1. Thus, the same argument with the connectedness of Σ gives the first inequality.

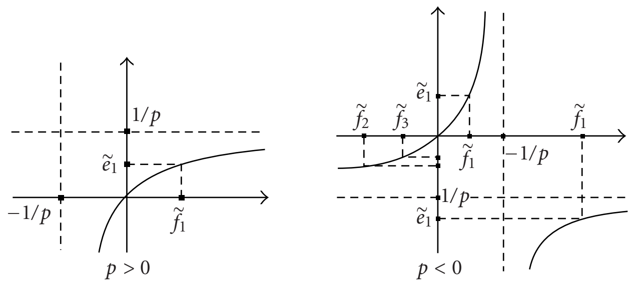

Figure 6.4

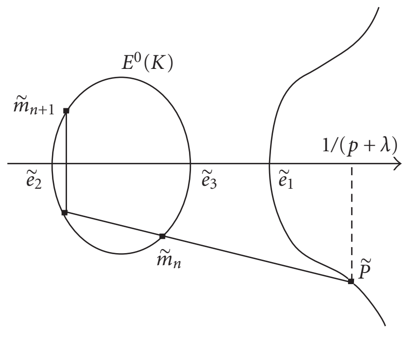

Figure 6.5

If $p > 0$, it is easy to see that $\widetilde{e}_1 > 0$. If $p = 0$, $\widetilde{e}_1 = \widetilde{f}_1 = -2f_1' > 0$. If $p < 0$, we look at the graph of the function $t \mapsto t/(1+pt)$ (see Figure 6.4): if $\widetilde{f}_1 > -1/p$, we obtain the inequalities $\widetilde{e}_1 < 1/p < \widetilde{e}_2 < \widetilde{e}_3 < 0$, which is impossible by Proposition 6.7; so $0 < \widetilde{f}_1 < -1/p$, and this gives us the result $\widetilde{e}_1 > 0$. $\square$

Remark 6.9. If $p > 0$, Figure 6.4 and Proposition 6.7 prove that we have $-1/p < \widetilde{f}_2 < \widetilde{f}_3 < 0 < \widetilde{f}_1$.

Propositions 6.5 and 6.7 and Lemma 6.8 give immediately the following important result.

Proposition 6.10. (1) *The cubic curve $E(K)$ has the form given in Figure 6.5.*

(2) *A solution (u_m) of difference equation (1.2) with $a > 0$ and $b+c+d > 0$ has minimal period n if and only if the point $\widetilde{P}$ is of order exactly n in the group $E(K)$.*

(3) *If a point M_0 of $Q^0(K)$ has minimal period n, it is also the case for every other point $M_0' \in Q^0(K)$.*

Now we can transform the cubic $E(K)$ (which is $\widetilde{\Gamma}(K)$ if $p(K) = 0$) by a last affine map τ_K, to get the canonical form of a regular cubic curve that we will parametrize by

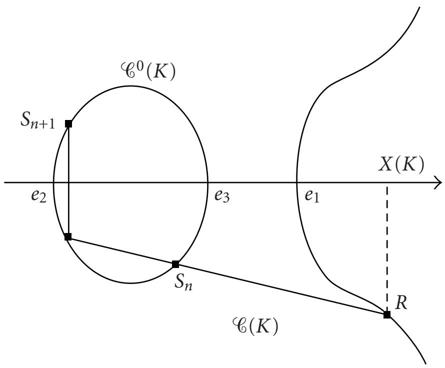

Figure 6.6

Weierstrass' function. We write (6.25) in the following form:

$$\frac{4\beta}{q}v^2 = 4u^3 - \frac{4(12p^2+8p(d+\lambda)+\gamma)}{q}u^2 + \frac{16(d+\lambda+3p)}{q}u - \frac{16}{q}. \tag{6.32}$$

We make an affinity on the variable v and then a translation on the variable u, by putting

$$y = 2\sqrt{\frac{\beta}{q}}v, \qquad x = u + t(K), \tag{6.33}$$

where

$$t(K) = -\frac{(12p^2+8p(d+\lambda)+\gamma)}{3q}. \tag{6.34}$$

Thus we obtain a new regular cubic curve $\mathscr{C}(K)$ in the standard Weierstrass' form:

$$y^2 = 4(x-e_1)(x-e_3)(x-e_2), \quad \text{with } e_i = \tilde{e}_i + t(K),\ e_2 < e_3 < e_1,\ \sum e_i = 0. \tag{6.35}$$

The point $\widetilde{P}$ becomes a point R, and the sequence S_n of iterates of a point S_0 in the bounded component $\mathscr{C}^0(K)$ of $\mathscr{C}(K)$ is the sequence $S_0 + nR$ for the natural group law on $\mathscr{C}(K)$ (see Figure 6.6). We denote the x-coordinate of R by

$$X(K) = \frac{1}{p+\lambda} + t(K), \tag{6.36}$$

and we will use the fact that its y-coordinate is negative (Lemma 6.8).

6.3. Parametrization with Weierstrass' function $\wp$ and the conjugated rotation. We can now parametrize this cubic by a Weierstrass function $\wp$, and obtain a group isomorphism of $\mathscr{C}(K)$ (real and complex points) with the torus $\mathbb{T}\times\mathbb{T}$.

There exist two positive numbers ω and ω' (which depend on b, c, d, and K) with the following property: if Λ is the lattice of $\mathbb{C}$ defined by $\Lambda = \{2n\omega + 2mi\omega' \mid (n,m) \in \mathbb{Z}^2\}$,

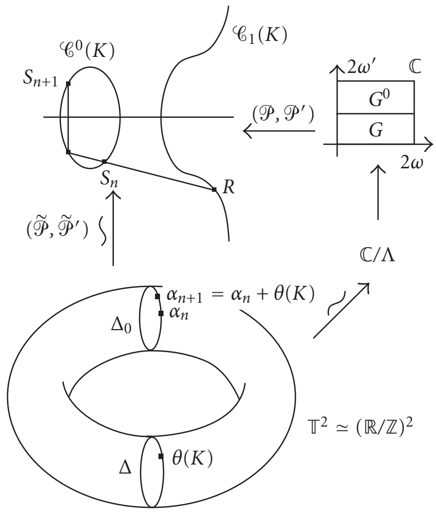

Figure 6.7

then the Weierstrass' elliptic function (depending on b, c, d and K)

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right] \tag{6.37}$$

is doubly periodic (its periods are the points of the lattice Λ), and gives the following parametrization of the entire cubic $\mathscr{C}(K)$ (its real and complex points):

$$x = \wp(z), \qquad y = \wp'(z). \tag{6.38}$$

The main properties of this parametrization are the following facts (see [1]):

(1) it transforms the addition on $\mathbb{C}$ into the addition on the cubic;

(2) it passes to quotient into a homeomorphism of topological groups from $\mathbb{T}^2 \approx \mathbb{C}/\Lambda$ onto $\mathscr{C}(K)$, which sends the circle $\Delta = \mathbb{T} \times \{1\}$ on the real connected unbounded component $\mathscr{C}^1(K)$ of the cubic with its point at infinity (which is so a subgroup), and the circle $\Delta_0 = \mathbb{T} \times \{-1\}$ on its real connected bounded component $\mathscr{C}^0(K)$, on which the real connected unbounded component acts then by adding;

(3) $(\wp, \wp')$ is one-to-one from the real segment $]0, \omega[$ onto the real unbounded branch of the cubic whose points have negative y-coordinates;

(4) one has the relations $e_1 = \wp(\omega)$, $e_2 = \wp(i\omega')$, and $e_3 = \wp(\omega + i\omega')$.

So it is easy to show that the addition of the point R to a point $S \in \mathscr{C}^0(K)$ is conjugated in $\mathbb{T} \times \mathbb{T}$ to the map $(e^{2i\pi\alpha}, -1) \mapsto (e^{2i\pi(\alpha+\theta)}, -1)$, for some well-defined number $\theta = \theta(K) \in]0, 1/2[$. For details, see [1, 3]. We have then proved the following theorem.

Theorem 6.11. *If $a>0$, then for every $K>K_m$ there exists a well-defined number $\theta(K)\in\,]0,1/2[$ such that the restriction of the map F of (1.3) to the curve $Q^0(K)$ is conjugated to the rotation on the circle with angle $2\pi\theta(K)\in\,]0,\pi[$ (see Figure 6.7).*

If ξ_K is the map from $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ onto the segment $G^0=i\omega'+[0,2\omega]$ of $\mathbb{C}:\exp(2i\pi u)\mapsto 2\omega u+i\omega'$, the homeomorphism of $\mathbb{T}$ onto $Q^0(K)$ is $\phi_k^{-1}\circ\kappa^{-1}\circ\psi_K^{-1}\circ\tau_K^{-1}\circ(\wp,\wp')\circ\xi_K$.

The result of the theorem was conjectured in [2] for the Lyness' difference equation $u_{n+2}u_n=u_{n+1}+a$, and it is proved in this case in [3, 4] for the more general case of difference equations $u_{n+2}u_n=\psi(u_{n+1})$ related to conics or cubic curves.

From Theorem 6.11 we see that, in accordance with rationality or irrationality of $\theta(K)$, the orbit of a point $M_0\in Q^0(K)$ will be periodic or dense in $Q^0(K)$. So it is essential to find the behavior of the solutions (u_n) of difference equation (1.2) to determine the image of the function $K\mapsto\theta(K)$ on $]K_m,+\infty[$. The only tool we have to make this is to find the limits of $\theta(K)$ when $K\to+\infty$ and $K\to K_m$.

We put

$$\varepsilon=\frac{e_1-e_3}{e_1-e_2}=\frac{\widetilde{e}_1-\widetilde{e}_3}{\widetilde{e}_1-\widetilde{e}_2},\qquad \nu=X(K)-e_1=\frac{1}{p+\lambda}-\widetilde{e}_1. \tag{6.39}$$

It is then a classical result about elliptic functions (see [1, 3] where all the calculations are made) that we have the formula

$$2\theta(K)=\frac{\int_0^{\sqrt{(e_1-e_2)/\nu}}\left(du/\sqrt{(1+u^2)(1+\varepsilon u^2)}\right)}{\int_0^{+\infty}\left(du/\sqrt{(1+u^2)(1+\varepsilon u^2)}\right)}. \tag{6.40}$$

By the same method as in [3], one sees that the function $K\mapsto\theta(K)$ is analytic on $]K_m,+\infty[$. Now we investigate the limits of this function when $K\to+\infty$ and $K\to K_m$.

6.4. The limits of $\theta(K)$ when K tends to $+\infty$ or K_m. We begin with the limit when $K\to+\infty$.

(1) The limit of $\theta(K)$ when $K\to+\infty$.

Proposition 6.12. *For $b+c+d>0$ (and $a=1$),*

$$\begin{gathered}\lim_{K\to+\infty}\theta(K)=\frac{1}{4}\quad \text{if } c>0,\\ \lim_{K\to+\infty}\theta(K)=\frac{1}{3}\quad \text{if } c=0,\ bd>0,\\ \lim_{K\to+\infty}\theta(K)=\frac{3}{8}\quad \text{if } c=0,\ bd=0 \text{ with } b+d>0.\end{gathered} \tag{6.41}$$

Proof. We know that $\lambda\to 0$ when $K\to+\infty$. So we will take λ as the variable, and find asymptotic expansion of the other parameters: $K,\alpha,\beta,p,f_i,\widetilde{e}_i$, and finally the parameters of integrals in (6.40): ε,ν.

We have first from (6.1) for $t=-\lambda$

$$K=\frac{1}{\lambda^2}\left(1-2b\lambda+2c\lambda^2-2d\lambda^3+\lambda^4\right), \tag{6.42}$$

and it follows easily from (6.42) asymptotic expansion of α and β, which gives

$$p = b\lambda^2 + (2b^2 - 4c)\lambda^3 + (d - 12bc + 4b^3)\lambda^4 + (8b^4 - 32b^2c + 16c^2 + 2bd)\lambda^5 + o(\lambda^5). \tag{6.43}$$

Remark that with condition $b + c + d > 0$ this expansion is not trivial.

From (6.1), whose solution $-\lambda$ and f_2 tend to 0, we obtain, with symmetric functions of solutions and relation (6.42), that f_i are solutions of

$$X^3 + (2d - \lambda)X^2 - \frac{1 - 2b\lambda}{\lambda^2}X + \frac{1}{\lambda} = 0, \tag{6.44}$$

or

$$X = \frac{\lambda}{1 - 2b\lambda}\left(1 + \lambda(2d - \lambda)X^2 + \lambda X^3\right). \tag{6.45}$$

We deduce of this relation a first expansion of $f_2 : f_2 = \lambda + 2b\lambda^2 + 4b^2\lambda^3 + o(\lambda^3)$, and then the expansion at order 5:

$$f_2 = \lambda\left(1 + 2b\lambda + 4b^2\lambda^2 + (2d + 8b^3)\lambda^3 + (16b^4 + 12bd)\lambda^4 + o(\lambda^4)\right). \tag{6.46}$$

We use simpler expansions for f_1 and f_3 coming from Lemma 6.1 and easy calculation:

$$f_1 \sim -\frac{1}{\lambda}, \qquad f_3 \sim \frac{1}{\lambda}. \tag{6.47}$$

Now we use formulas $\widetilde{e}_i = 2/(\lambda - f_i + 2p)$ to get expansions of $\widetilde{e}_i$:

$$\begin{gathered} \widetilde{e}_1 \sim 2\lambda, \qquad \widetilde{e}_3 \sim -2\lambda, \\ \widetilde{e}_2 = \frac{2}{-8c\lambda^3 - 24bc\lambda^4 + (32c^2 - 64b^2c - 8bd)\lambda^5 + o(\lambda^5)} \end{gathered} \tag{6.48}$$

(the case $c = 0$ needs expansion of f_2, and thus of p, until order 5, if $bd > 0$).

Now, easy calculation gives

$$\varepsilon = 16c\lambda^4 + 24bc\lambda^5 + (64b^2c + 8bd - 32c^2)\lambda^6 + o(\lambda^6), \tag{6.49}$$

$$\widetilde{e}_1 - \widetilde{e}_2 \sim -\frac{2}{-8c\lambda^3 - 24bc\lambda^4 + (32c^2 - 64b^2c - 8bd)\lambda^5 + o(\lambda^5)}, \qquad \nu \sim \frac{1}{\lambda}. \tag{6.50}$$

So we obtain

$$\sqrt{\frac{\widetilde{e}_1 - \widetilde{e}_2}{\nu}} = \frac{1}{\sqrt{4c\lambda^2 + 12bc\lambda^3 + (32b^2c + 4bd - 16c^2)\lambda^4 + o(\lambda^4)}}. \tag{6.51}$$

If $c > 0$, relations (6.49) and (6.51) give $\lambda \sim A\varepsilon^{1/4}$ and thus $\sqrt{(\widetilde{e}_1 - \widetilde{e}_2)/\nu} \sim B/\varepsilon^{1/4}$. If $c = 0$ but $bd > 0$, the same relations give $\lambda \sim A'\varepsilon^{1/6}$ and thus $\sqrt{(\widetilde{e}_1 - \widetilde{e}_2)/\nu} \sim B'/\varepsilon^{1/3}$. With [3, Lemma 4] for the integrals of formula (6.40) we find the two first limits of Proposition 6.12.

If $c = 0$ and $bd = 0$, with $b+d > 0$, all the terms of the previous asymptotic development (6.48) of $2/\widetilde{e}_2$ are zero, so we must make more tedious calculations, in the two cases $c = b = 0$, $d > 0$, and $c = d = 0$, $b > 0$. We find that the asymptotic development is $-8b^2\lambda^7 + o(\lambda^7)$ in the first case, and $-8d^2\lambda^7 + o(\lambda^7)$ in the second one. So we obtain easily the same limit 3/8 of $\theta(K)$ at infinity, in these two cases. □

Remark 6.13. If $c = 0$ and $bd = 1$, we have $u_{n+2}u_n = (1+bu_{n+1})/(du_{n+1}+u_{n+1}^2) = 1/du_{n+1}$, and the sequence satisfies $u_{n+2}u_{n+1}u_n = 1/d$. Thus it is 3-periodic for every (u_1, u_0), that is for every $K > K_m$. This fact is in accordance with the relation $\lim_{K\to+\infty}\theta(K) = 1/3$.

If $b = d = 0$, $a = 1$, $c > 0$, we have $u_{n+2}u_n = (1+cu_{n+1}^2)/(c+u_{n+1}^2)$ and for $c = 1$ we get the difference equation $u_{n+2}u_n = 1$, whose every solution is 4-periodic, which is compatible with the relation $\lim_{K\to+\infty}\theta(K) = 1/4$.

Corollary 6.14. *Under hypothesis (3.3), the function $K \mapsto \theta(K)$ is constant if and only if* 3 *(if $c = 0$ and $bd > 0$),* 8 *(if $c = 0$ and $bd = 0$), or* 4 *(if $c > 0$) is a common minimal period to all the nonconstant solutions of difference equation (1.2).*

Proof. If n is a common minimal period, then $\theta(K) = r(K)/n$, an irreducible fraction. By continuity, r, and thus θ, is constant. Conversely, if θ is a constant θ_0, this number is the limit of $\theta(K)$ when $K \to +\infty$, so $\theta \equiv 1/4$, 1/3, or 3/8, and thus all the solutions of (1.2) have common period 3, 4, or 8. □

(2) The limit of $\theta(K)$ when $K \to K_m$, if $b+d > 0$.

We have a general result in the case $b+d > 0$.

Proposition 6.15. *Define the number p_0 as*

$$p_0 := \lim_{K\to K_m} p(K) = \frac{d\lambda_0^2 - 4c\lambda_0 + b}{K_m + 2c + 2d\lambda_0 - 2\lambda_0^2}. \tag{6.52}$$

With hypothesis

$$b+d > 0, \tag{6.53}$$

it holds that

$$\ell - f_0 > 0, \qquad f_0 + \lambda_0 < 0, \qquad p_0 + \lambda_0 > 0, \qquad \lambda_0 - \ell + 2p_0 < 0, \tag{6.54}$$

$$\lim_{K\to K_m}\theta(K) = \frac{1}{\pi}\tan^{-1}\sqrt{\frac{2(\ell - f_0)(p_0+\lambda_0)}{(f_0+\lambda_0)(\lambda_0 - \ell + 2p_0)}} \in \left]0, \frac{1}{2}\right[. \tag{6.55}$$

Proof. We have $f_0 + \lambda_0 = -2\sqrt{(d+\ell)^2 - 1/\ell^2} \le 0$, and this quantity is zero only if $d+\ell = 1/\ell$. With relation (3.1), one deduce in this case that $b = d = 0$. So under hypothesis (6.53) we have $f_0 + \lambda_0 < 0$.

Then $\ell - f_0 = 2\ell + d + \sqrt{(d+\ell)^2 - 1/\ell^2} > 2\ell > 0$.

If $p < 0$ for some $K > K_m$, then $\widetilde{e}_2 < 0$ (see Figure 6.4). If $p > 0$ for some $K > K_m$, then the remark after the proof of Lemma 6.8 and Figure 6.4 show that $\widetilde{e}_2 < 0$. If $p = 0$ for some $K > K_m$, then $\tilde{e}_2 = 2/(\lambda - f_2) < 0$ if $b+d>0$ (Lemma 6.1). In the three cases, $\tilde{e}_2 = 2/(\lambda - f_2 + 2p) < 0$, so we have

$$\frac{2}{\tilde{e}_2} = \lambda - f_2 + 2p < 0 \quad (\text{if } b+d>0), \tag{6.56}$$

and thus, by limit, $\lambda_0 - \ell + 2p_0 \le 0$. We know also that $p+\lambda > 0$, and thus $p_0 + \lambda_0 \ge 0$. We will see that these two numbers are never zero if $b+d>0$.

Proof that $p_0+\lambda_0>0$. The denominator of $p_0+\lambda_0$ is $\beta_0 = \lim_{K\to K_m}\beta(K) \ge 4c + 3d\ell + b/\ell > 0$ if $b+d>0$. Some tedious calculation, using formulas (3.1) in the form $\ell^2 - 1/\ell^2 = b/\ell - d\ell$, and (6.4) in the form $1/\lambda_0 = \ell^2(d+\ell+\sqrt{d^2+d\ell+b/\ell})$, gives for the numerator N of $p_0+\lambda_0$ the formula (up to a possible positive factor)

$$N = b^2 - d^2 + \left(b\ell^2 + 4\ell + d\right)\sqrt{d^2 + d\ell + \frac{b}{\ell}}. \tag{6.57}$$

But $d\ell + b/\ell > 0$ if $b+d>0$. Thus we have $N > b^2 - d^2 + (d)\sqrt{d^2} = b^2 \ge 0$, and $N>0$.

Proof that $\lambda_0 - \ell + 2p_0 < 0$. If $b+d>0$, then by formula (6.4), $f_0 \neq -\lambda_0$, that is, $-\lambda_0$ is a simple root of the equation $G(t,t) = K_m$. Thus, $(d/dt)G(t,t)|_{t=-\lambda_0} > 0$. So we have

$$\lambda_0 - \lambda = A(K-K_m) + o(K-K_m) \tag{6.58}$$

for some constant $A>0$ (see Figure 6.1). But ℓ is exactly a double root of the equation $G(t,t) = K_m$, thus we have

$$\ell - f_2 = B\sqrt{K-K_m} + o\left(\sqrt{K-K_m}\right) \tag{6.59}$$

for some constant $B>0$. So it is easy to see that

$$p - p_0 = O(K-K_m). \tag{6.60}$$

Now, we know that $\lambda - f_2 + 2p < 0$. Suppose that $\lambda_0 - \ell + 2p_0 = 0$. We write

$$\lambda - f_2 + 2p = (\lambda - \ell + 2p) + (\ell - f_2) = (\lambda - \ell + 2p) - (\lambda_0 - \ell + 2p_0) + (\ell - f_2) \tag{6.61}$$

and by (6.58), (6.59), and (6.60) this is equal to

$$\lambda - \lambda_0 + 2(p-p_0) + (\ell - f_2) = B\sqrt{K-K_m} + o\left(\sqrt{K-K_m}\right), \tag{6.62}$$

which is positive if $K - K_m$ is sufficiently small. But this contradicts the relation (6.56), and thus the relation $\lambda_0 - \ell + 2p_0 = 0$ is impossible.

Proof of formula (6.55). Now we have, when $K \to K_m$

$$\varepsilon = \frac{\tilde{e}_1 - \tilde{e}_3}{\tilde{e}_1 - \tilde{e}_2} = \frac{f_1 - f_3}{f_1 - f_2} \times \frac{\lambda - f_2 + 2p}{\lambda - f_3 + 2p} \longrightarrow \frac{f_0 - \ell}{f_0 - \ell} \times \frac{\lambda_0 - \ell + 2p_0}{\lambda_0 - \ell + 2p_0} = 1. \tag{6.63}$$

We have also, when $K \to K_m$

$$\frac{\widetilde{e}_1 - \widetilde{e}_2}{\nu} = \frac{\widetilde{e}_1 - \widetilde{e}_2}{1/(p+\lambda) - \widetilde{e}_1} \longrightarrow \frac{2(\ell - f_0)(p_0+\lambda_0)}{(f_0+\lambda_0)(\lambda_0 - \ell + 2p_0)}, \tag{6.64}$$

finite and positive number from (6.54). We can now determine easily the limits of the integrals in formula (6.40), and this gives (6.55). □

(3) The limit of $\theta(K)$ when $K \to K_m$, if $b = d = 0$.

In this case, the difference equation (1.2) becomes

$$u_{n+2}u_n = \frac{1+cu_{n+1}^2}{c+u_{n+1}^2}, \quad c > 0. \tag{6.65}$$

Proposition 6.16. *If $b = d = 0$, $c > 0$ (and $a = 1$), then*

$$\lim_{K\to K_m} \theta(K) = \frac{1}{\pi}\tan^{-1}\frac{1}{\sqrt{c}}. \tag{6.66}$$

Proof. An easy calculation shows that $\widetilde{e}_2$ and $\widetilde{e}_3 \to -1$ when $K \to K_m$, and that we have $\widetilde{e}_1 \sim_{K\to K_m} (c/(c+1))(1/(1-\lambda))$ ($\lambda \to 1$ when $K \to K_m$).

So we have $\lim_{K\to K_m} \varepsilon = 1$, $\widetilde{e}_1 - \widetilde{e}_2 \sim (c/(c+1))(1/(1-\lambda))$, and $\nu \sim (c^2/(c+1))(1/(1-\lambda))$. Thus we find $\sqrt{(\widetilde{e}_1 - \widetilde{e}_2)/\nu} \sim 1/\sqrt{c}$. So it is immediate to go to the limit in integrals of formula (6.40). We get $\lim_{K\to K_m} = (1/\pi)\tan^{-1}(1/\sqrt{c})$. □

We can now study the global behavior of the solutions of (1.2) when $a > 0$, and in particular we can study whether IPCB holds.

6.5. The global behavior of the solutions of difference equation (1.2) with $a = 1$. We have introduced in Section 2 the general property of IPCB. We will hope that it is a description of a possible behavior of the solutions of (1.2) when $a > 0$. This behavior holds in the case $a = 0$ with $b^2 \neq c^3$ or $bd \neq 2c^2$ (see Theorem 5.1).

More precisely, our goal is to determine if IPCB is true for some values of the parameters b, c, d, with the hope that for values of parameters for which IPCB is false all nonconstant solutions of (1.2) have a common minimal period (as in Lyness' case).

A useful tool to find whether the difference equation (1.2) (with $a = 1$) has IPCB is the following.

Lemma 6.17. (a) *If the function $K \mapsto \theta(K)$ is not constant on $]K_m, +\infty[$, then the difference equation (1.2) has IPCB.*

(b) *A sufficient condition for this is that* $\lim_{K\to+\infty}\theta(K) \neq \lim_{K\to+K_m}\theta(K)$, *or, if these limits are equal to $\theta_0 = p/q$, an irreducible fraction, that q cannot be a common minimal period to all the solutions of (1.2).*

Proof. Assertion (b) is obvious from corollary of Proposition 6.12. The proof of assertion (a) is the main result of [3], and we refer to this paper for the details. We will only precise an argument (to prove the sensitivity to initial conditions of points of the set B) which seems to be different in the case $b = d = 0$: the transformation from $Q^0(K)$ onto

$\mathscr{C}^0(K)$ passes through a point at infinity of $\Gamma(K)$ (in direction $x = y$, see Lemma 6.2). So the argument of [3, 4] about uniform convergence of $h_c^{-1}(K') \circ (\wp_{K'}, \wp'_{K'})^{-1}$ to $h_c^{-1}(K) \circ (\wp_K, \wp'_K)^{-1}$ when $K' \to K$ (see [3, Lemma 5]) does not work directly. But it is enough to compose ψ_K, κ, and ϕ_K to get the isomorphism of $Q^0(K)$ on $\mathscr{C}^0(K)$ in finite terms:

$$(X,Y) \longmapsto (x,y) = \left(2\sqrt{\frac{\beta}{2}}\frac{X-Y}{XY-\lambda^2-p(X+Y)}, t(K) - \frac{X+Y}{XY-\lambda^2-p(X+Y)}\right). \quad (6.67)$$

But if $b = d = 0$, then $p = -4c\lambda/\beta$, and the denominator in the formulas satisfies the inequalities (by the point (1) of Lemma 6.2)

$$XY - \lambda^2 - p(X+Y) \geq \frac{4c\lambda}{\beta}(X+Y) > 0 \quad \text{on } Q^0(K), \quad (6.68)$$

and so the argument of uniform convergence given in the proof of [3, Theorem 2] works also for the present situation. □

Now we put

$$H(b,c,d) := \frac{2(\ell - f_0)(p_0+\lambda_0)}{(f_0+\lambda_0)(\lambda_0 - \ell + 2p_0)}. \quad (6.69)$$

We have the following general but abstract result.

THEOREM 6.18. *Suppose that $b+d>0$.*

(a) *If $H(b,0,d) \neq 3$, then the difference equation (1.2) with $c = 0$, $bd > 0$ (and $a = 1$), has IPCB. If $H(b,0,0) \neq 3+2\sqrt{2}$, and if $H(0,0,d) \neq 3+2\sqrt{2}$, then (1.2) with $c = d = 0$, $b > 0$ and with $c = b = 0$, $d > 0$ (and $a = 1$) has IPCB.*

(b) *If $c > 0$ and $H(b,c,d) \neq 1$, then the difference equation (1.2) has IPCB.*

(c) *In every case, the* dichotomy property holds*: either (1.2) with $a > 0$ has IPCB or all its nonconstant solutions have a common minimal period which is* 3, 4, *or* 8.

Proof. If $c > 0$, Propositions 6.12 and 6.15 assert that

$$\lim_{K\to K_m} \theta(K) = \frac{1}{\pi}\tan^{-1}\sqrt{H(b,c,d)}, \qquad \lim_{K\to+\infty}\theta(K) = \frac{1}{4}, \quad (6.70)$$

so the first condition (b) of Lemma 6.17 holds if $H(b,c,d) \neq 1$. If $c = 0$ and $bd > 0$, $\lim_{K\to+\infty}\theta(K) = 1/3$, so the condition is that $\tan^{-1}\sqrt{H(b,0,d)} \neq \pi/3$, that is $H(b,0,d) \neq 3$. If $c = 0$ and $bd = 0$, then $\lim_{K\to+\infty}\theta(K) = 3/8$, so the first condition (b) of Lemma 6.17 holds if $H(b,0,d) \neq \tan^2(3\pi/8) = 3+2\sqrt{2}$.

Finally, point (c) follows from corollary of Proposition 6.12. □

Remark 6.19. The dichotomy property was proved for (1.2) with $a = 0$ in Theorem 5.1, with in this case the common minimal period 5.

It seems very difficult, in the general case, to identify the values of (b,c,d) for which conditions of Theorem 6.18 hold, in particular because ℓ is defined only implicitly by

(6.42): the formula giving $H(b,c,d)$ is very complicated. Moreover, two cases in Theorem 6.18 involves period 8, which is not easy to see. So we will study first some interesting special cases.

7. Global behavior of solutions of (1.2) in some particular cases with $a > 0$

We begin with the case $b = d > 0$ (and $a = 1$), which is the only case with $\ell = 1$.

7.1. The difference equation $u_{n+2}u_n = (1 + du_{n+1} + cu_{n+1}^2)/(c + du_{n+1} + u_{n+1}^2)$, $d > 0$.

Theorem 7.1. *Consider the difference equation*

$$u_{n+2}u_n = \frac{1 + du_{n+1} + cu_{n+1}^2}{c + du_{n+1} + u_{n+1}^2}, \quad d > 0. \tag{7.1}$$

(1) *If (u_n) is a solution of (7.1), then $(1/u_n)$ is also a solution.*
(2) *The following formula holds*

$$H(d,c,d) = \frac{d+2}{2c+d}. \tag{7.2}$$

(3) *If $0 < c \neq 1$, the difference equation (7.1) has IPCB.*
If $c = 1$, all the nonconstant solutions of (7.1) have the common minimal period 4.
Conversely, if there exists $(u_1, u_0) \neq (\ell, \ell)$ with period 4*, then $c = 1$ and all nonconstant solutions of (7.1) have minimal period* 4 *in common.*
(4) *If $c = 0$, the difference equation (1.2) becomes*

$$u_{n+2}u_{n+1}u_n = \frac{1 + du_{n+1}}{d + u_{n+1}}. \tag{7.3}$$

If $d \neq 1$, then (7.3) has IPCB.
If $d = 1$, then all nonconstant solutions of (7.3) have the common minimal period 3.
Conversely, if there exists $(u_1, u_0) \neq (\ell, \ell)$ with period 3*, then $d = 1$ and all nonconstant solutions of (7.3) have minimal period* 3 *in common.*

Proof. Point (1) is obvious. Point (2) follows from very tedious calculations to identify the four factors of formula (6.69) and to get simplifications which give (7.2).

Now suppose $c > 0$. Then (7.1) has IPCB if $H(d,c,d) \neq 1$, that is $c \neq 1$ (Lemma 6.17). If $c = 1$, (7.1) becomes $u_{n+2}u_n = 1$, all of whose solutions are 4-periodic. Conversely, suppose that $(u_1, u_0) \neq (\ell, \ell)$ has period 4. We have

$$u_{n+2}u_n - 1 = (1-c)\frac{1 - u_{n+1}^2}{c + du_{n+1} + u_{n+1}^2}, \qquad u_{n-2}u_n - 1 = (1-c)\frac{1 - u_{n-1}^2}{c + du_{n-1} + u_{n-1}^2}, \tag{7.4}$$

and these two quantities are equal if $u_{n+2} = u_{n-2}$ (period 4). So the function $t \mapsto (1 - t^2)/(c + dt + t^2)$ takes the same value at u_{n+1} and u_{n-1}. This function is obviously decreasing on $[0, +\infty[$, and thus $u_{n+1} = u_{n-1}$ for every $n : u_n$ is 2-periodic. But it is easy to prove that for difference equation (1.2) a 2-periodic solution is constant. Thus $u_1 = u_0 = \ell$, which contradicts our hypothesis.

Now suppose $c = 0$. Then $H(d,0,d) = 1 + 2/d \neq 3$ if and only if $d \neq 1$. So, by Lemma 6.17, IPCB holds if $d \neq 1$. If $d = 1$, then $H(1,0,1) = 3$. But in this case solutions of (7.3) are obviously 3-periodic. Conversely, suppose that $(u_1, u_0) \neq (\ell, \ell)$ is 3-periodic, so u_n is nonconstant. Then formula (7.3) shows that the function $t \mapsto (1 + dt)/(d + t)$ takes the same value for at least two distinct values of (u_n). So the homographic function $t \mapsto (1 + dt)/(d + t)$ is constant, that is $d = 1$; and thus all the solutions of (7.3) are 3-periodic. □

If $0 < c \neq 1$ and $d = c + 1$, point (3) of Theorem 7.1 gives the behavior of a particular case of the difference equation of [5], because the factor $1 + u_{n+1}$ appears both in the numerator and the denominator of (7.1).

Corollary 7.2. *The particular case of difference equation (4.1):*

$$u_{n+2}u_n = \frac{1 + cu_{n+1}}{c + u_{n+1}}, \quad 0 < c \neq 1, \tag{7.5}$$

has IPCB.

Remark 7.3. It is easy to see that there is another case of difference equation (4.1) which has IPCB:

$$u_{n+2}u_n = \frac{\alpha u_{n+1} + \alpha^3/\delta^3}{u_{n+1} + \delta}. \tag{7.6}$$

7.2. The difference equations $u_{n+2}u_{n+1}u_n = b + 1/u_{n+1}$ and $u_{n+2}u_{n+1}u_n = 1/(d + u_{n+1})$. These cases are $c = d = 0$ and $b = c = 0$. First, we remark that by setting $u_n = 1/v_n$ and inverting b and d, these two difference equations reduce one to the other. So we prove the following result only for the first one.

Theorem 7.4. *The difference equations*

$$u_{n+2}u_{n+1}u_n = b + \frac{1}{u_{n+1}}, \quad b > 0, \qquad u_{n+2}u_{n+1}u_n = \frac{1}{d + u_{n+1}}, \quad d > 0, \tag{7.7}$$

have IPCB.

Proof. We look only at the first equation. Some calculations give

$$H(b,0,0) = \frac{4\ell^3 - b}{b} = 3 + \frac{4}{\ell^4 - 1} = 3 + \frac{4}{\ell b}. \tag{7.8}$$

The condition $H(b,0,0) \neq 3 + 2\sqrt{2}$ gives $b \neq b_0 = \sqrt{2}(\sqrt{2} - 1)^{1/4} \approx 1.1345433$. In this case, the difference equation (7.7) has IPCB. □

If $b = b_0$, the two limits of $\theta(K)$ at $+\infty$ and at K_m are the same: 3/8. If θ was constant, its value would be 3/8, and so all the nonconstant solutions of (7.7) would have minimal period 8. But this is not the case: for $b = b_0$, $u_0 = 10$, $u_1 = 1$, computer calculation gives $u_8 \approx 8.269$ and $u_9 \approx 1.23$, with a deviation from (10,1) less than the numerical error due to the error on digits of b_0, as proved by easy calculation. By the dichotomy, and property (7.7) has IPCB.

7.3. The difference equation $u_{n+2}u_n = (1+cu_{n+1}^2)/(c+u_{n+1}^2)$**.** This is the case when $b = d = 0$, $c > 0$ (and $a = 1$).

Theorem 7.5. *Consider the difference equation*

$$u_{n+2}u_n = \frac{1+cu_{n+1}^2}{c+u_{n+1}^2}, \quad c > 0. \tag{7.9}$$

If (u_n) *is solution of (6.65), then* $(1/u_n)$ *is also solution of (6.65).*

If $c \neq 1$, *then (6.65) has IPCB. If* $c = 1$, *all nonconstant solutions of (6.65) have minimal period* 4.

Conversely, if there exists $(u_1, u_0) \neq (\ell, \ell)$ *with period* 4, *then* $c = 1$ *and all the nonconstant solutions of (6.65) have common minimal period* 4.

Proof. The first point is obvious. We have $\lim_{K\to+\infty}\theta(K) = 1/4$ and $H(0,c,0) = 1/c$ (Proposition 6.16). So if $c \neq 1$, the function θ is nonconstant, and by Lemma 6.17 difference equation (6.65) has IPCB.

If $c = 1$, $u_{n+2}u_n = 1$, and then all nonconstant solutions are 4-periodic (and 2 is never a period, see the proof of Theorem 7.1).

Now, suppose that $(u_1, u_0) \neq (\ell, \ell)$ is 4-periodic. In particular $(u_3, u_2) = (u_{-1}, u_{-2})$. If we put $X = u_1^2$ and $Y = u_0^2$, this equality is equivalent to

$$(1-c^2)\left[(1+cX)^2 - Y^2(c+X)^2\right] = 0, \qquad (1-c^2)\left[(1+cY)^2 - X^2(c+Y)^2\right] = 0. \tag{7.10}$$

If $c \neq 1$, easy calculations give $X^2 = Y^2 = 1 = \ell$, and so $u_1 = u_0 = \ell$, which is false. So we have $c = 1$. □

7.4. The difference equation $u_{n+2}u_{n+1}u_n = (1+bu_{n+1})/(d+u_{n+1})$**.** It happens when $c = 0$ and $b, d > 0$. In this case, we have not a simple form for $H(b,0,d)$, but however we can determine the behavior of the solutions (we have already done it if $b = d$ in point (4) of Theorem 7.1, with the aid of form (7.2) of $H(d,c,d)$).

Theorem 7.6. *Let the difference equation*

$$u_{n+2}u_{n+1}u_n = \frac{1+bu_{n+1}}{d+u_{n+1}}, \quad b, d > 0. \tag{7.11}$$

If $bd \neq 1$, *then (7.11) has IPCB.*

If $bd = 1$, *then all the nonconstant solutions of (7.11) have the common period* 3.

Conversely, if there exists $(u_1, u_0) \neq (\ell, \ell)$ *which is* 3*-periodic, then* $bd = 1$ *and all the nonconstant solutions of (7.11) have the common period* 3.

Proof. Of course, if $bd = 1$, then $u_{n+2}u_{n+1}u_n = 1/d$, and so u_n is 3-periodic. If there exists $(u_1, u_0) \neq (\ell, \ell)$ which is 3-periodic, u_n is nonconstant, and $u_{n+2}u_{n+1}u_n$ is constant, and thus the function $t \mapsto (1+bt)/(d+t)$ has the same value for at least two distinct values of the sequence (u_n); so this homographic function is constant, that is $bd = 1$.

If $bd \neq 1$, then function θ is nonconstant (otherwise $\theta \equiv \lim_{K\to+\infty}\theta(K) = 1/3$, and 3 is a common period, thus $bd = 1$). Thus, by Lemma 6.17, (7.11) has IPCB. □

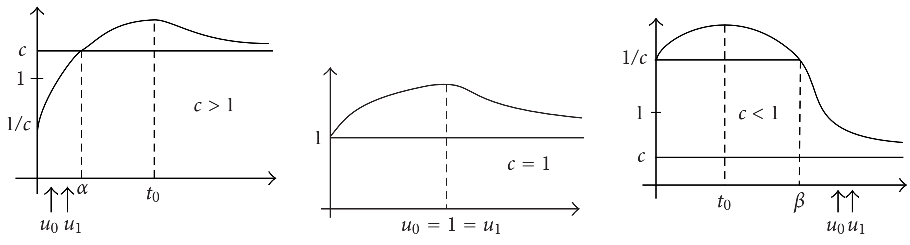

Figure 7.1

Remark 7.7. (1) If $b = d$, we find again point (4) of Theorem 7.1.

(2) If (u_n) is a solution of (7.11), then $(1/u_n)$ is a solution of (7.11) where b and d are inverted.

7.5. The equations $u_{n+2}u_n = (1+bu_{n+1}+cu_{n+1}^2)/(c+u_{n+1}^2)$ **and** $u_{n+2}u_n = (1+cu_{n+1}^2)/(c+du_{n+1}+u_{n+1}^2)$**.** The first one is the case when $b > 0$, $c > 0$, and $d = 0$. But if we put $v_n = 1/u_n$ and invert b and d, the second equation becomes the first one. So we study only this difference equation.

Theorem 7.8. *The difference equations*

$$u_{n+2}u_n = \frac{1+bu_{n+1}+cu_{n+1}^2}{c+u_{n+1}^2}, \qquad u_{n+2}u_n = \frac{1+cu_{n+1}^2}{c+du_{n+1}+u_{n+1}^2} \tag{7.12}$$

have IPCB.

Proof. We only give the proof for the first equation (7.12). Easy calculation gives

$$H(b,c,0) = \frac{3b\ell+4}{\ell(4c\ell^2+b)}. \tag{7.13}$$

If $H(b,c,0) \neq 1$, then (7.12) has IPCB. If $H(b,c,0) = 1$, then (7.12) has IPCB if 4 is not a common minimal period of all its nonconstant solutions. Indeed it is the case.

If a solution is 4-periodic, we have $u_{n+2}u_n = u_nu_{n-2}$, and thus the function

$$g(t) = \frac{1+bt+ct^2}{c+t^2} \tag{7.14}$$

takes the same value at $t = u_{n+1}$ and $t = u_{n-1}$, for every n.

The three possible forms of the graph of g are given in Figure 7.1. If $c > 1$, we choose u_1 and u_0 distinct in $]0,\alpha[$; then u_n would be 2-periodic nonconstant and have minimal period 4, which is impossible. If $c < 1$, we would have the same contradiction with u_1 and u_0 distinct in $]\beta,+\infty[$. If $c = 1$, $u_1 = u_0 = 1$ would generate a constant solution, which is impossible, for $\ell \neq 1$. $\square$

8. The general case of (1.2) with $a = 1$, $b \neq d$, and $b, c, d > 0$

We know by the dichotomy property that in the case $c > 0$ and $a > 0$, either (1.2) has IPCB or all its nonconstant solution have common minimal period 4. So, we suppose that all solutions have period 4, and we try to obtain a contradiction for a great set of values for the parameters b, c, d. We work as in the proof of Theorem 7.8: if u_n is 4-periodic, then the function

$$g(t) := \frac{1 + bt + ct^2}{c + dt + t^2} \tag{8.1}$$

takes the same value at u_{n+1} and at u_{n-1}. So, we look at the eventual property (8.2) of g:

$$\begin{gathered} \text{there exists } t_0 > 0 \text{ such that } t_0 \neq \ell, (g^{-1} \circ g)(t_0) = \{t_0\} \\ \text{or there exist } \varnothing \neq \,]\alpha, \beta[\, \subset \mathbb{R}_*^+ \text{ such that } (g^{-1} \circ g)(]\alpha, \beta[) = \,]\alpha, \beta[. \end{gathered} \tag{8.2}$$

A particular case of the second condition holds if g is one-to-one. Property (8.2) of g will be the essential tool for proving IPCB for some values of b, c, d.

Lemma 8.1. *If, for $bcd > 0$, function g has property (8.2), then (1.2) (with $a = 1$) has IPCB.*

Proof. If g satisfies the second assertion of (8.2), and all nonconstant solutions have minimal period 4, we choose $u_0 \neq u_1$ both in $]\alpha, \beta[$; thus $u_{n+1} = u_{n-1}$, and u_n is 2-periodic nonconstant, which is a contradiction; by dichotomy property, (1.2) has IPCB.

If g satisfies the first part of property (8.2), we take $u_1 = u_0 = t_0$. If the generated sequence is 4-periodic, it is constant and different from ℓ, and this is impossible. So 4 is not a common period of all solutions, and (1.2) has IPCB. □

We will now study function g. The numerator $N(g')$ of g' is

$$(cd - b)t^2 + 2(c^2 - 1)t + bc - d. \tag{8.3}$$

(a) We begin with the case $c = 1$. Then $N(g')(t) = (d - b)(t^2 - 1)$. In accordance with the sign of $b - d$, g has on $[0, +\infty[$ a strict maximum or a strict minimum at $t_0 = 1$. So if $t > 0$ and $g(t) = g(t_0)$ we have $t = t_0$. But $\ell \neq 1$, for $b \neq d$, and g satisfies (8.2): (1.2) has IPCB.

(b) Now, suppose that $c \neq 1$ and $cd = b$. Then $N(g')(t) = 2(c^2 - 1)t + bc - d$, and $bc - d \neq 0$ (otherwise we would have $b = d$). The root of $N(g')$ is $t_1 = (bc - d)/2(1 - c^2) = -d/2 < 0$, and g is injective on $[0, +\infty[$; so (1.2) has IPCB.

(c) The case when $c \neq 1$ and $bc = d$ is analogous: the roots of $N(g')$ are 0 and $-2/b$, and thus g is one-to-one on $]0, +\infty[$.

(d) Now suppose that $(bc - d)(cd - b) > 0$. Then all the roots of $N(g')$ have the same sign. It is easy to see that if $c > 1$, then $bc - d > 0$, and then $g'(0) > 0$ and $g(0) < c$, thus g' has (at least) one root on $] - \infty, 0[$, and then two, and property (8.2) holds: (1.2) has IPCB. If $c < 1$, we make the same reasoning, *mutatis mutandis*.

(e) Finally, we look at the case when $(bc - d)(cd - b) < 0$. So $N(g')$ has a negative and a positive roots.

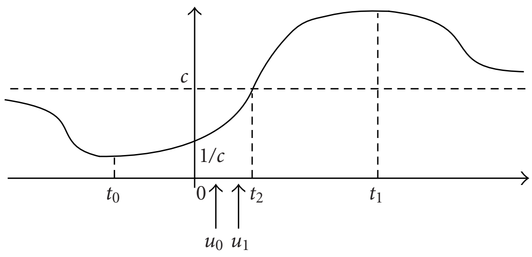

Figure 8.1

Table 8.1

Set of parameters	Global behavior	$H(b,c,d)$
(1) $c = d = 0,\ b > 0$ [or $c = b = 0,\ d > 0$]	IPCB	$H(b,0,0) = 3 + \dfrac{4}{\ell^4 - 1}$
(2) $b = d > 0,\ c > 0$	IPCB if $c \neq 1$ 4 common period if $c = 1$	$H(d,c,d) = \dfrac{d+2}{2c+d}$
(3) $b = d > 0,\ c = 0$	IPCB if $d \neq 1$ 3 common period if $d = 1$	$H(d,0,d) = 1 + \dfrac{2}{d}$
(4) $b = d = 0,\ c > 0$	IPCB if $c \neq 1$ 4 common period if $c = 1$	$H(0,c,0) = \dfrac{1}{c}$
(5) $bd > 0,\ c = 0, b \neq d$	IPCB if $bd \neq 1$ 3 common period if $bd = 1$	?
(6) $bc > 0,\ d = 0$ [or $cd > 0,\ b = 0$]	IPCB	$H(b,c,0) = \dfrac{3b\ell + 4}{\ell(4c\ell^2 + b)}$
(7) $bcd > 0,\ b \neq d$	IPCB	?

If $c > 1$, we look at the two cases $bc - d > 0$ and $bc - d < 0$. In the first case, $g'(0) > 0$, and the roots t_0, t_1 of $N(g')$ satisfy $t_0 < 0 < t_2 := (t_0 + t_1)/2 = (1 - c^2)/(cd - b) < t_1$. So $g \geq c$ on $[t_2, +\infty[$, and the interval $]0, t_2[$ satisfies obviously the second condition of property (8.2), and (1.2) has IPCB (see Figure 8.1).

If $bc - d < 0$, then $g'(0) < 0$, and $t_0 < t_2 < 0 < t_1$; we define t_3 by $g(t_3) = g(0) = 1/c$, that is $t_3 = (bc - d)/(1 - c^2) > t_2$. So it is obvious that the interval $]t_3, +\infty[$ satisfies the second condition of (8.2), and then (1.2) has IPCB.

If $c < 1$, we make the same reasoning, *mutatis mutandis*.

So we have proved our final result on the global behavior of the solutions of (1.2).

Theorem 8.2. *If $a, b, c, d > 0$ and $b \neq d$, then (1.2) has IPCB.*

We can summarize all our results for the behavior of solutions of (1.2) when $a = 1$ in Table 8.1.

Table 9.1

Set of parameters	Set J	Minimal periods of some solutions
(1) $c = d = 0$, $b > 0$ [or $c = b = 0$, $d > 0$]	$]1/3, 1/2[\setminus\{3/8\}$	3, 4 not minimal periods 5, 7, 9, every $k \geq 11$ min. periods 6, 8, 10?
(2) $b = d > 0$, $c > 0$	$]0,1/2[$	every $k \geq 3$
(3) $b = d > 0$, $c = 0$	$]1/4,1/2[$	4 not minimal period 3, 5 and every $k \geq 7$ min. periods 6 ?
(4) $b = d = 0$, $c > 0$	$]0,1/2[$	every $k \geq 3$
(5) $bd > 0$, $c = 0$, $b \neq d$	$]1/4,1/2[$	4 not minimal period 3, 5 and every $k \geq 7$ min. periods 6 ?
(6) $bc > 0$, $d = 0$ [or $cd > 0$, $b = 0$]	$]0, 1/2[\setminus\{1/4\}$	every $k \geq 3$
(7) $bcd > 0$, $b \neq d$	$]0,1/2[$	every $k \geq 3$

9. Determination of possible minimal periods in some particular cases with $a > 0$

In this part, we will investigate, for each row of the above table, what are the possible periods of solutions of (1.2) (with $a = 1$) associated to the corresponding set of values of parameters. Precisely, we will say that "an integer k is minimal period" if there exist values (b,c,d) in this set of parameters of the table and an initial point (u_1, u_0) such that the associated solution of (1.2) has *minimal* period k.

Theorem 9.1. *If $a = 1$, the possible minimal periods of solutions of (1.2), for each set of parameters of the previous table, are given in Table 9.1.*

Proof. In fact, the only difficult case is the first row of the table; rows (2), (4), and (6) are easy, and the case of row (3) will be deduced from row (1). The case of rows (5) and (7) will follow from rows (2) and (3).

The essential tool is the determination of the range of the function θ, or at least a set J which is an open interval, except perhaps a point, which is included in this image, and then to find what are the integers k such that there exists an integer q relatively prime with k satisfying $q/k \in J$. For obtaining such a set J, we will use for given b, c, d the limits of $\theta(K)$ when $K \to +\infty$ and $K \to K_m$, and study the union of all intervals with as extremities these two limits when (b,c,d) varies in the authorized domain. The method we will use is in [3], but we recall it briefly.

With the formula for the limits of function θ, a set J is easy to find in the following cases:

$(*)$ $J =]0,1/2[$ for rows (2) and (4);

$(*)$ $J =]0,1/2[\setminus\{1/4\}$ for row (6);

$(*)$ $J =]1/4, 1/2[$ for row (3);

$(*)$ $J =]1/3, 1/2[\setminus\{3/8\}$ for row (1).

Arguments of continuity and connexity prove easily that in rows (5) and (7) same J as in rows (3) and (2) work.

Now it is obvious that every integer $k \geq 3$ is a minimal period in the case of rows (2), (4), and (7).

For row (6), every integer $k \geq 3$, $k \neq 4$, is a minimal period. The proof of Theorem 7.8 shows only that 4 is not a common minimal period to all nonconstant solutions of (1.2). But it is easy to see that $(u_1, u_0) = (1,1)$ is 4-periodic for $b = 9/4$ and $c = 5/4$. So $J =]0, 1/2[$ in fact, and 4 is a minimal period.

If we suppose the result for the first row of the table is true, then for row (3) we have only to test integers 3, 4, 6, 8, 10, because the set J is bigger for row (3) than for row (1). But 1/3, 3/8, and 3/10 are in $]1/4, 1/2[$, and so 3, 8, and 10 are minimal periods (for 3, this follows also from Theorem 7.1 which asserts that $1/3 \in J$).

For integer 4, if a solution would be 4-periodic, we would have $g(u_{n+1}) = g(u_{n-1})$, where $g(t) = (1 + dt)/t(d + t)$. But g is one-to-one on $]0, +\infty[$, so u_n would be 2-periodic, which is impossible.

So only the case of row (1) needs a proof.

In order to find minimal periods in this case, we search first for fractions $q/n \in J$ with q a prime number which does not divide n. We use an improvement of the prime number theorem due to Rosser and Schoenfeld (see [11]), in the weak following form: if $n \geq 52$, one has

$$\frac{n}{\ln n} \leq \pi(n) \leq \left(1 + \frac{3}{2\ln n}\right)\frac{n}{\ln n}, \tag{9.1}$$

where $\pi(n)$ is the cardinal of the set of prime numbers not greater than n.

We use also an optimal majorization for the cardinal $\omega(n)$ of the set of distinct prime factors of the integer n (see [10]):

$$\omega(n) \leq 1.38402\frac{\ln n}{\ln(\ln n)}. \tag{9.2}$$

So, if $n \geq 52$, the cardinal of the set of integers q relatively prime to n such that $q/n \in]1/3, 1/2[$ is not empty (and n is a minimal period) if the function

$$f(x) := \frac{x/2 - 1}{\ln(x/2 - 1)} - \frac{x/3 - 1}{\ln(x/3 - 1)}\left(1 + \frac{3}{2\ln(x/3)}\right) - 1.38402\frac{\ln x}{\ln(\ln x)} - 1 \tag{9.3}$$

is positive. Computer calculations give $f(780) < 0$, $f(k) > 0$ for k integer, $781 \leq k \leq 10^5$.

Then it is not difficult to see that for $x > 10^5$ one has $f(x) \geq g_\mu(x) := \mu(x/\ln x)$ for μ sufficiently small ($\mu = 0.3$ works), and this inequality proves that $f(x) > 0$ for $x > 10^5$. Thus we can conclude that every $n \geq 781$ is a minimal period of some solution of (7.7) for some $b > 0$.

For the integer $n \leq 780$ we use the method of [3]: we prove that the intervals [528,780], [360,527], [256,359], [180,255], [124,179], [88,123], [64,87], [48,63], [36,47], [28,35], and [24,27] contain only minimal periods (we use the prime numbers 263, 179, 127, 89, 61, 43, 31, 23, 17, 13, and 11 for these intervals, for more details, see [3]).

Then, we test directly each integer n in [3,23] to find, if possible, an integer q relatively prime to n such that $q/n \in]1/3, 1/2[$.

Finally, this method gives the results in row (1). $\square$

Remark 9.2. In case of row (6) ($a = 1$, $d = 0$, $bc > 0$), one can see that 4 is minimal period if and only if $c > 1$ and $K = b^2/(c^2 - 1)$. For example, if $(u_1, u_0) = (1, 1)$, one has $u_2 = u_3 = c + \sqrt{c^2 - 1}$, and u_n is 4-periodic. More generally, for every $u_1, u_0 > 0$ and $c > 1$, (u_1, u_0) is 4-periodic for one (and only one) value of $b > 0$.

One can see also that if $a - 1$, 3 is a minimal period for $K > K_m$ if and only if one has $c(K + c) = 1 - bd$. And so it exists (u_1, u_0) which is 3-periodic if and only if $((1 - bd)/c) - c > K_m$.

Final note. We have recently found a 1996 preprint of E. C. Zeeman: “Geometric unfolding of a difference equation.” It seems that this paper was not published. In this paper, Zeeman obtains the results of our paper [3] on Lyness’ difference equation. His methods are the same as ours, but more qualitative, and allow to solve some open problems from [3, 4] and the present paper. We will publish these solutions in a foregoing work about other difference equations.

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